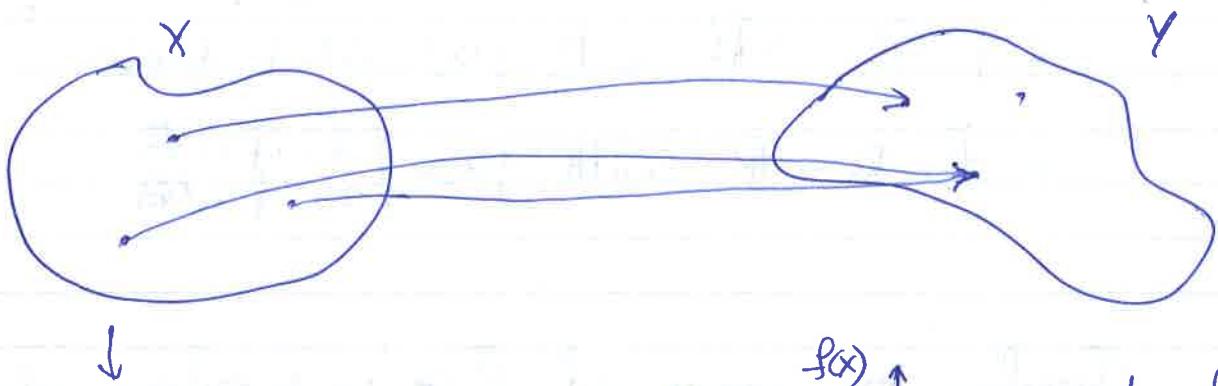


(1)

Functions - Limits :

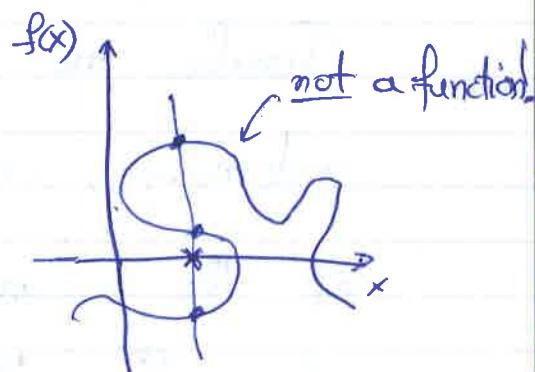
→ Def: Let X, Y be two non-empty sets.

A function $f: X \rightarrow Y$ is a relation that associates every $x \in X$ with an element $y \in Y$.



↓
there should be a single arrow coming out of each element of X , for f to be a function. i.e., the same $x \in X$ cannot be sent to two distinct elements of Y .

(But of course two different elements of X can be sent to the same element of Y).



X is called the domain of f ,
 Y is the codomain (or target space) of f .
 and $f(X) = \{f(x) : x \in X\}$ the image of f .

(2)

→ Def: A real function is any function with codomain \mathbb{R} (i.e., any function that takes values in \mathbb{R} ; the domain can be anything).

- ex:
- $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = c^{\mathbb{R}}$ $\forall x \in \mathbb{R}$; a constant function.
 - $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = 2x^2 + 1$ $\forall x \in \mathbb{R}$.
 - $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$.

Clearly, the image of f is a subset of the codomain; and it can of course be a strict subset of the codomain (for example, for $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = c^{\mathbb{R}}$ $\forall x \in \mathbb{R}$, $f(\mathbb{R}) = \{c\} \subsetneq \mathbb{R}$).

→ Def: The function $f: X \rightarrow Y$ is called onto if $f(X) = Y$ (i.e., the image equals the codomain).

I.e., f is onto $\iff \forall y \in Y$, there exists $x \in X$ with $f(x) = y$.

(3)

⚠ Of course, there may exist $y \in Y$ such that

there exist more than one $x \in X$ sent to y through f
(i.e., more than one $x \in X$ with $f(x)=y$).

ex: • $f: \mathbb{R} \rightarrow \mathbb{R}$, with $f(x)=x^2$, $\forall x \in \mathbb{R}$:

this is not onto, as $\nexists x \in \mathbb{R}$ st. $x^2 = -1$
($x^2 \geq 0 \quad \forall x \in \mathbb{R}$).

• $f: \mathbb{R} \rightarrow [0, +\infty)$, with $f(x)=x^2$, $\forall x \in \mathbb{R}$:

this is onto;

let $y \in [0, +\infty)$. We want to show that

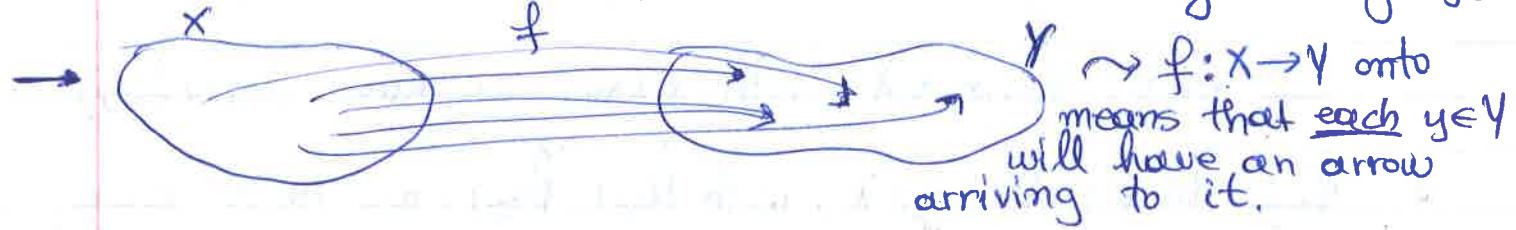
$\exists x \in \mathbb{R}$ st. $y = f(x)$.

Indeed, we know that there exists a unique

$x \geq 0$ st. $x^2 = y$; the element $\sqrt{y} \in \mathbb{R}$. So, $y = f(\sqrt{y})$.

(Note that also $f(-\sqrt{y})=y$, so there exist

two elements of the domain sent to y through f)



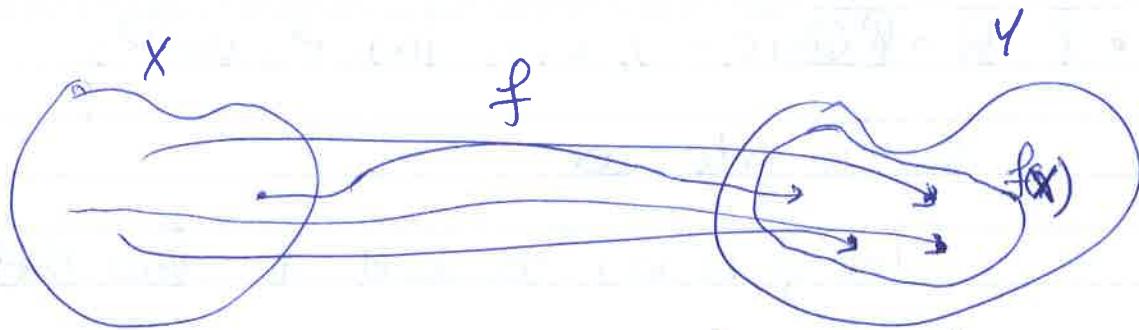
(4)

→ Def: The function $f: X \rightarrow Y$ is called 1-1

if distinct elements of X are sent through f to distinct elements of Y ,

i.e. if each $y \in f(X)$ cannot be the image of more than one element of X ,

i.e. if $\forall y \in f(X)$ there exists a unique $x \in X$ s.t. $y = f(x)$.



$f: X \rightarrow Y$ onto means that each $y \in f(X)$ has a unique arrow arriving at it.

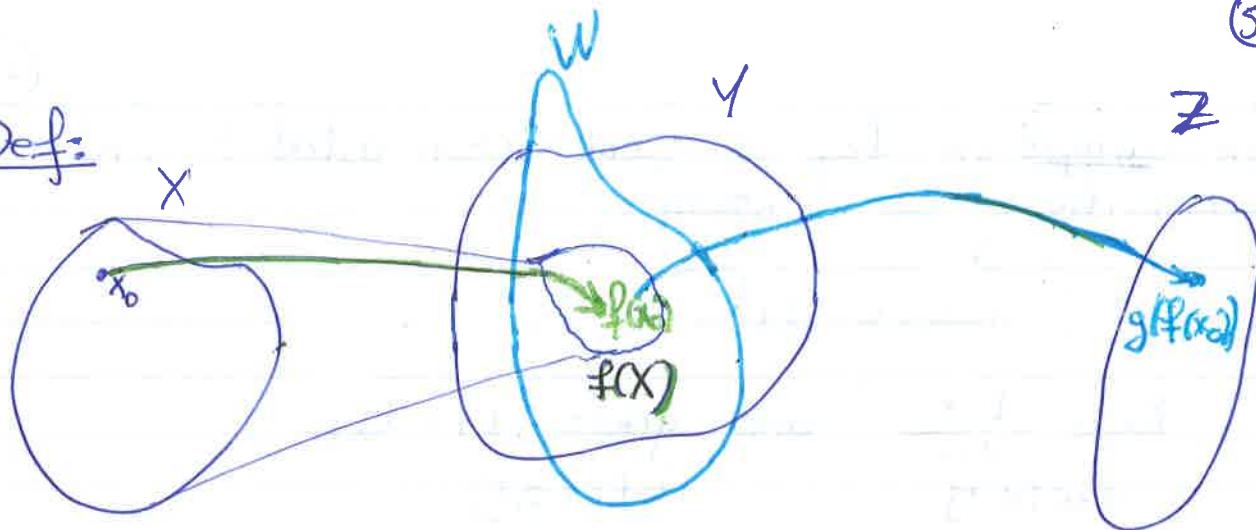


In practice, to show that $f: X \rightarrow Y$ is 1-1, we can do any of the following two (equivalent) things:

- Show that: $\forall x_1, x_2 \in X$, with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.
- Show that: $\forall x_1, x_2 \in X$, with $f(x_1) = f(x_2)$, we have $x_1 = x_2$.

(5)

→ Def:



Let $f: X \rightarrow Y$ and $g: W \rightarrow Z$ be two functions,
such that $f(X) \subseteq W$,

We define ~~gof~~ : $X \rightarrow Z$ by
 \downarrow
 $gof(x) = g(f(x)),$
 $\forall x \in X.$
 the composition
 of f with g

This is a well-defined function, as, $\forall x \in X$:

- ① $f(x)$ is unique (as f is a function),
- ② $f(x) \in f(X) \subseteq W$, so $g(f(x))$ is well-defined.
- ③ $g(f(x))$ is unique (as g is a function).

(2)

→ An example on how we work when asked to find compositions of functions:

→ Let $f, g : [0, 1] \rightarrow [0, 1]$, with

$$f(x) = \frac{1-x}{1+x} \quad \text{and} \quad g(t) = 4t(1-t)$$

$$\forall x \in [0, 1] \quad \forall t \in [0, 1]$$

- (i) Show that f, g are well-defined.
- (ii) find $f \circ g$ and $g \circ f$, if they exist.
- (iii) Are $f \circ g$ and $g \circ f$ $1-1$?

:

(i) Since each element of $[0, 1]$ is sent to a unique element of \mathbb{R} via f and via g , to show that f, g are well-defined we just need to show that $[0, 1]$ can indeed be a codomain of f and g , i.e. that

$$f([0, 1]) \subseteq [0, 1] \quad \text{and} \quad g([0, 1]) \subseteq [0, 1].$$

Indeed: • $f([0, 1]) \subseteq [0, 1]$: Let $x \in [0, 1]$.

We need: $f(x) \in [0, 1] \Leftrightarrow 0 \leq f(x) \leq 1$

$\Leftrightarrow 0 \leq \frac{1-x}{1+x} \leq 1$, which is true (check it!)

• $g([0, 1]) \subseteq [0, 1]$: Let $t \in [0, 1]$.

(7)

$$\text{We need: } g(t) \in [0, 1] \Leftrightarrow 4t(1-t) \in [0, 1] \Leftrightarrow 0 \leq 4t(1-t) \leq 1.$$

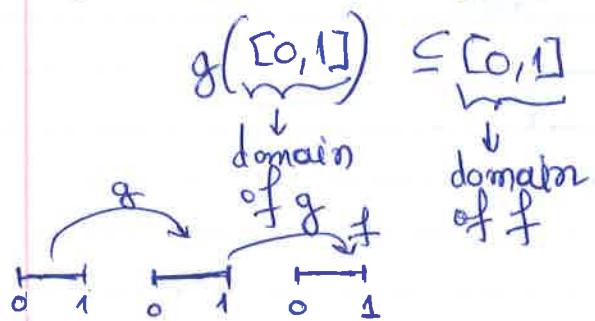
Since $t \geq 0$, $1-t \geq 0$ and $4 \geq 0$, we have: $4t(1-t) \geq 0$.

$$\begin{aligned} \text{And } 4t(1-t) \leq 1 &\Leftrightarrow 4t - 4t^2 - 1 \leq 0 \Leftrightarrow 4t^2 - 4t + 1 \geq 0 \\ &\Leftrightarrow (2t)^2 - 2 \cdot 2t + 1 \geq 0 \\ &\Leftrightarrow (2t - 1)^2 \geq 0, \end{aligned}$$

which is true.

$$\text{So, indeed } 0 \leq 4t(1-t) \leq 1.$$

(ii) $f \circ g$: $f \circ g$ is well defined if



We have seen in (i) that, indeed, $\forall t \in [0, 1], g(t) \in [0, 1]$.

So, $g([0, 1]) \subseteq [0, 1]$, so $f \circ g$ is well-defined.

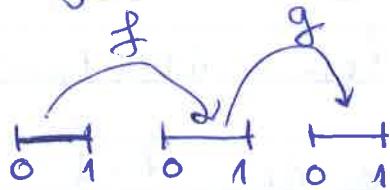
$$\text{And: } f(g(t)) = f(4t(1-t)) = \frac{1 - 4t(1-t)}{1 + 4t(1-t)}, \forall t \in [0, 1].$$

So, $f \circ g : [0, 1] \rightarrow [0, 1]$,

$$\text{with } f \circ g(t) = \frac{1 - 4t(1-t)}{1 + 4t(1-t)}, \forall t \in [0, 1].$$

(8)

- gof : gof is well defined if:



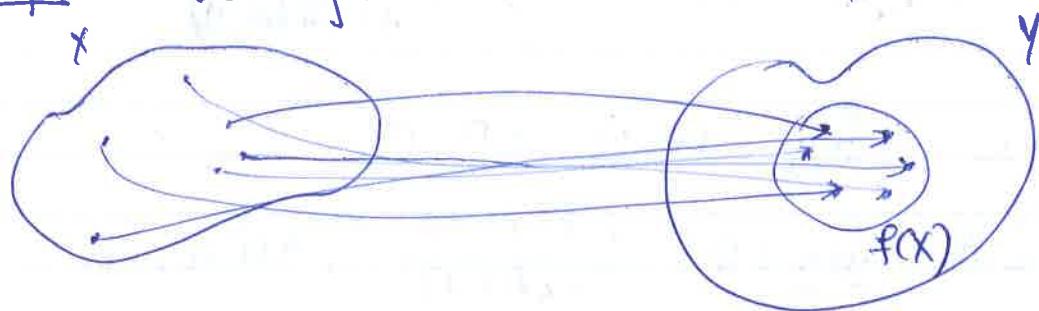
$f([0,1]) \subseteq [0,1]$, which is true
 domain of f domain of g
 (by (i))

So, gof is well defined, and

$$gof : [0,1] \rightarrow [0,1]$$

$$\begin{aligned} gof(t) &= g(f(t)) = \\ &= 4f(t)(1-f(t)) = \\ &= 4 \cdot \frac{1-t}{1+t} \cdot \left(1 - \frac{1-t}{1+t}\right) \end{aligned}$$

→ Def: Let $f: X \rightarrow Y$ be 1-1.



Consider $f: X \rightarrow f(X)$. This is 1-1 and onto.

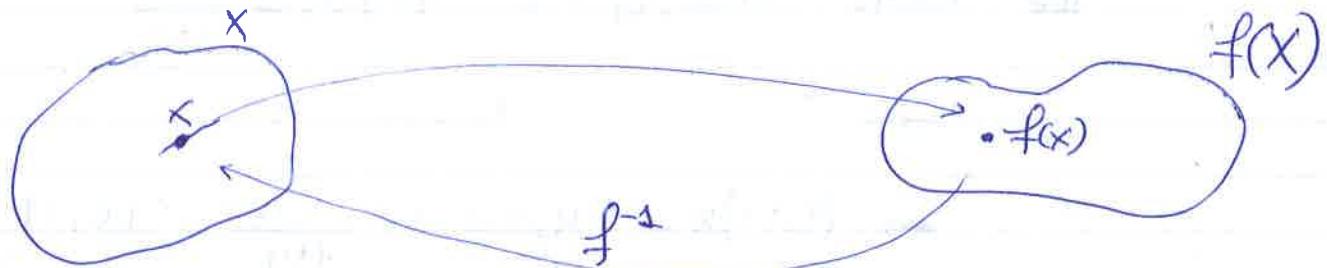
That is, $\forall y \in f(X)$ there exists a unique $x \in X$ with $f(x) = y$.

(9)

the inverse of f !

We define $f^{-1}: f(X) \rightarrow X$, such that

$\forall y \in f(X)$, $f^{-1}(y) = x$, where x is the unique element of X with $f(x) = y$.



$f^{-1} \circ f: X \rightarrow X$, $f^{-1} \circ f(x) = f^{-1}(f(x)) = x$, $\forall x \in X$.)

$f \circ f^{-1}: f(X) \rightarrow f(X)$, $f \circ f^{-1}(y) = y$, $\forall y \in f(X)$.)

identity maps!

ex: Find f^{-1} , g^{-1} in the example above.

First of all, we need to check if these functions exist, i.e. if f and g are 1-1.

- for f : Let $x_1, x_2 \in [0, 1]$ with $f(x_1) = f(x_2)$

$$\Rightarrow \frac{1-x_1}{1+x_1} = \frac{1-x_2}{1+x_2} \Rightarrow (1-x_1)(1+x_2) = (1+x_1)(1-x_2)$$

$$\Rightarrow 1+x_2-x_1-x_1x_2 = 1-x_2+x_1-x_1x_2 \Rightarrow 2x_1 = 2x_2 \Rightarrow$$

$$\Rightarrow x_1 = x_2, \text{ so } f \text{ is 1-1.}$$

So, $f^{-1}: f([0, 1]) \rightarrow [0, 1]$ is well-defined.

Let's find it:

(10)

$f([0,1]) = ?$ We notice that f is onto.

Indeed, let $y \in [0,1]$. We will show that $y = f(x)$, for some $x \in [0,1]$:

We want $x \in [0,1]$ s.t. $y = f(x) = \frac{1-x}{1+x}$

$$\Leftrightarrow 1-x = y + yx \Leftrightarrow$$

$$\Leftrightarrow (y+1)x = 1-y \Leftrightarrow x = \frac{1-y}{1+y} \in [0,1],$$

$\begin{matrix} | \\ 1+y \neq 0 \end{matrix}$

as $y \in [0,1]$.

So, $f^{-1} : [0,1] \rightarrow [0,1]$

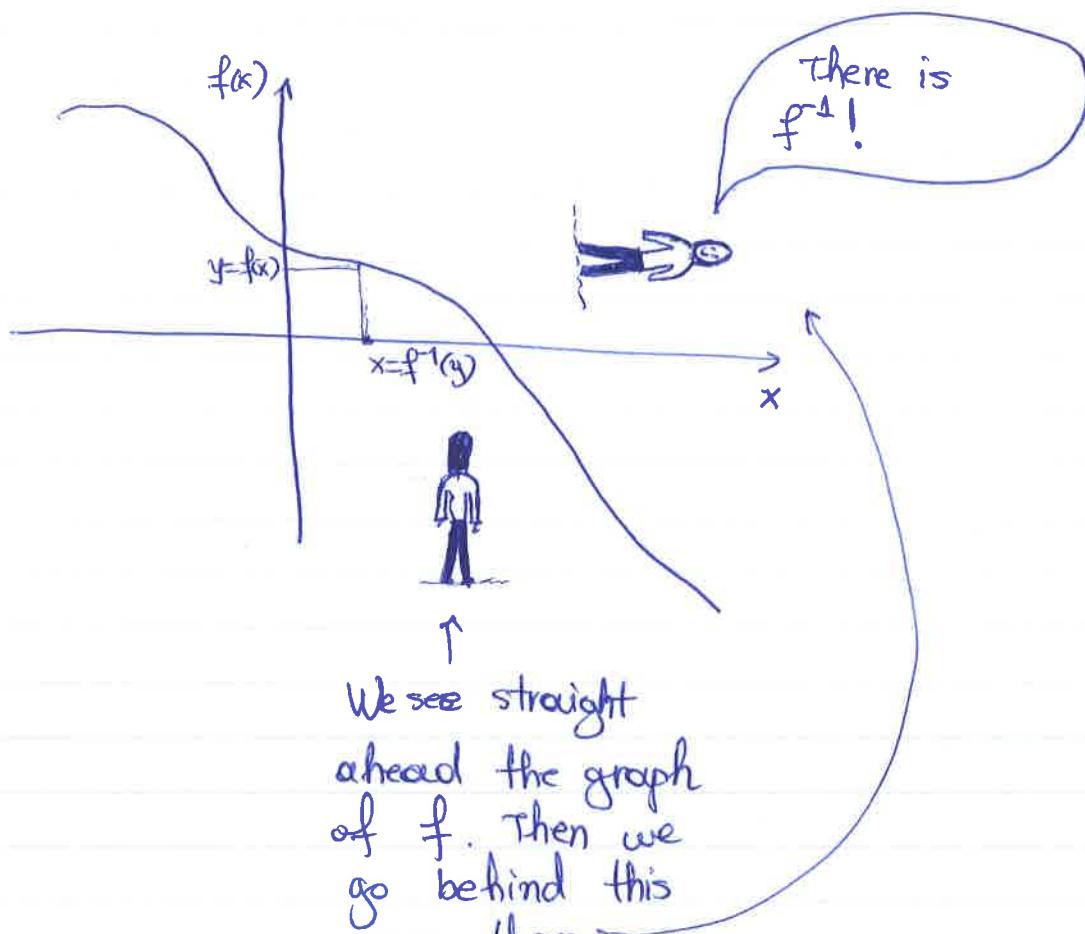
with $f^{-1}(y) = \frac{1-y}{1+y}$, $\forall y \in [0,1]$.

Notice that $f^{-1} \circ f = f$, i.e. $f \circ f$ is the identity map from $[0,1]$ to $[0,1]$.
not a general truth!

- for g : g is not $1-1$: $g(0) = g(1) (= 0)$.

So, g^{-1} is not well-defined.

How to visualise the inverse function:



We see straight ahead the graph of f . Then we go behind this paper, there, and turn around to face it. What we see is the graph of f^{-1}

(see how, due to our change of perspective, the old Oy becomes our "new Ox ")

6

Limits of functions:

→ Def: Let $A \subseteq \mathbb{R}$. A point $x_0 \in \mathbb{R}$ is an

x_0 doesn't have to be in A to be an accumulation point of A!

accumulation point of A if
in any neighbourhood of x_0

"arbitrarily close to x ", I can find elements of
A different from x ".

I.e., if: $\forall \delta > 0$, there exists $x_0 \in A$ in the neighbourhood $(x_0 - \delta, x_0 + \delta)$,

i.e., if : $\forall \delta > 0$, there exists $x^* \neq x_0$ in A,
 with $|x - x_0| < \delta$

i.e. if: $\exists \delta > 0$, $(x_0 - \delta, x_0 + \delta) \cap (A \setminus \{x_0\}) \neq \emptyset$

Ex: ④ For $A = \{0\} \subseteq \mathbb{R}$, 0 is not an accumulation point.

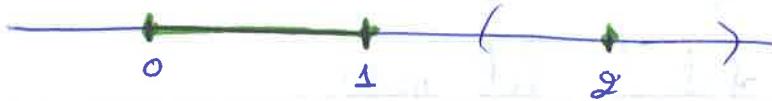
point of A :



in fact, no matter which neighbourhood of o we consider, we cannot find elements of A different from o in that neighbourhood.

(2)

(2)



$$A = [0, 1] \cup \{2\}.$$

All elements of $[0, 1]$ are accumulation points of A , but 2 is not an accumulation point of A :

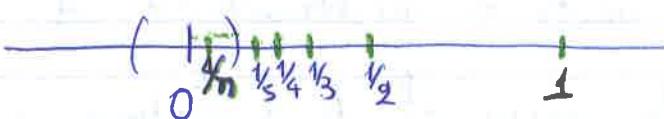
for instance, consider the neighbourhood $(2 - \frac{1}{2}, 2 + \frac{1}{2})$ of 2 . This neighbourhood doesn't contain any elements of A other than 2 .

(3)



$A = (0, 1)$. All elements of $\underbrace{[0, 1]}$ are accumulation points of A .
contains points that are not in A !

(4)

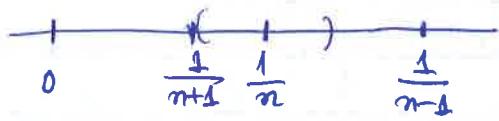


$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. 0 is an accumulation point of A (even though $0 \notin A$).

Indeed: Let $\delta > 0$. There exists $n_0 \in \mathbb{N}$ s.t. $0 < \frac{1}{n_0} < \delta$, so $\frac{1}{n_0} \in (0 - \delta, 0 + \delta)$; and $\frac{1}{n_0} \in A \setminus \{0\}$.

(3)

Notice that $\frac{1}{n}$ is not an accumulation point of A, for any $n \in \mathbb{N}$. Indeed, let $n \in \mathbb{N}$.



Let $\delta = \frac{1}{n} - \frac{1}{n+1}$. There is no element of $A \setminus \{\frac{1}{n}\}$ in the neighbourhood $(\frac{1}{n} - \delta, \frac{1}{n} + \delta)$ (i.e., if $x \in A$ and $|x - \frac{1}{n}| < \delta$, then $x = \frac{1}{n}$), so $\frac{1}{n}$ is not an accumulation point of A.

⑤ $A = \mathbb{Q}$. All elements of \mathbb{R} are accumulation points of A.

→ Def: Let $A \subseteq \mathbb{R}$, and $x_0 \in A$. \triangleleft Not just in \mathbb{R} .

We say that x_0 is an isolated point of A

if it is not an accumulation point of A.

I.e. if : there exists a neighbourhood of x_0 ,

such that the only point of A it contains is x_0 ,

i.e. if : $\exists \delta > 0$ s.t. $(x_0 - \delta, x_0 + \delta) \cap A = \{x_0\}$.

(4)

Ex: In examples ①, ② and ④ above, the isolated points of A are 0, 2 and $\frac{1}{n}$ then,

respectively.

→ We now want to define limits of functions; we want to explain what we will mean by $\lim_{x \rightarrow x_0} f(x) = l (\in \mathbb{R})$.

Let $f: A^{\text{SAR}} \rightarrow \mathbb{R}$, and x_0 an accumulation point of A.
 I want a definition of $\lim_{x \rightarrow x_0} f(x) = l$ that will allow me to say that:

"No matter how close to l I look at, $f(x)$ will be at least that close to l when $x \neq x_0$ is close enough to x_0 ".

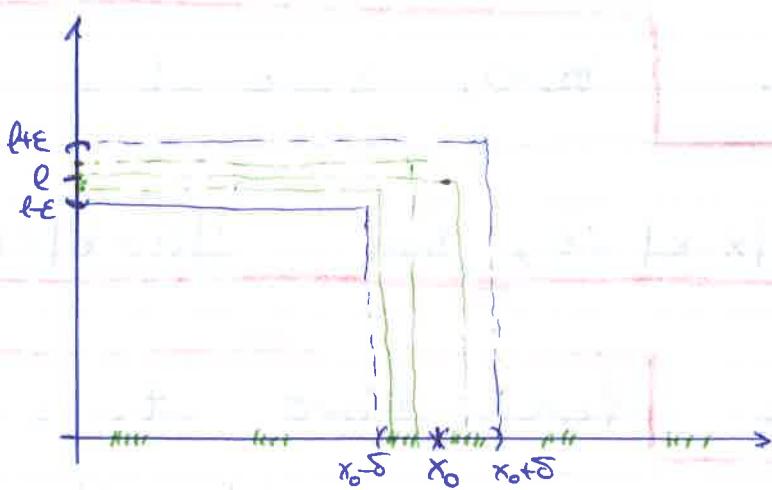
I.e.: "No matter how close to l I look at
 (i.e., no matter how small a neighbourhood $(l-\varepsilon, l+\varepsilon)$ of l I consider),

there exists some (potentially tiny) "punctured" neighbourhood $(x_0-\delta, x_0+\delta) \setminus \{x_0\}$ of x_0 , s.t.

all the elements of A in that punctured neighbourhood

(5)

are sent to $(l-\epsilon, l+\epsilon)$ via f ".



!
i.e. maybe $f(x_0)$
is not even
defined; that's OK!

notice that x_0 does not have to be in A , and we don't care whether $f(x_0) \in (l-\epsilon, l+\epsilon)$ or not. But we want x_0 to be an accumulation point of f , so that there actually exist elements of A in $(x_0-\delta, x_0+\delta) \setminus \{x_0\}$.

So, we are led to the following definition:

→ Def: Let $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, and x_0 be an accumulation point of A .

We say that "the limit of f as x goes to x_0 equals l "

and we write

$$\boxed{\lim_{x \rightarrow x_0} f(x) = l \text{ if: } \forall \epsilon > 0, \exists \delta > 0 \text{ s.t.: } f(A \cap ((x_0 - \delta, x_0 + \delta) \setminus \{x_0\})) \subseteq (l - \epsilon, l + \epsilon).}$$

(6)

This is the same as saying :

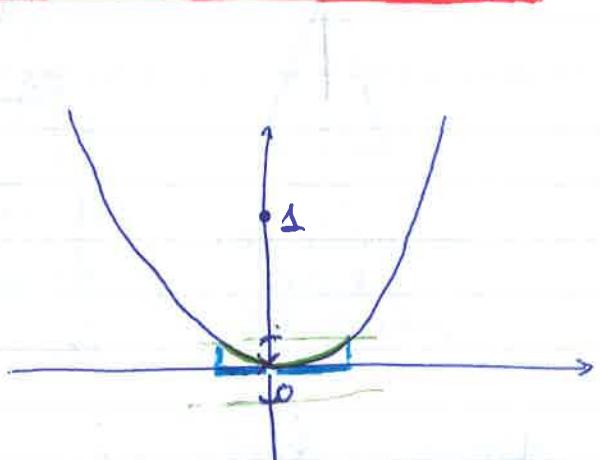
$$\lim_{x \rightarrow x_0} f(x) = l \text{ if : } \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. :}$$

$$\text{if } x \in A, x \neq x_0 \text{ and } |x - x_0| < \delta, \text{ then } |f(x) - l| < \epsilon,$$

$$\text{or } \lim_{x \rightarrow x_0} f(x) = l \text{ if : } \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. :}$$

$$\text{if } x \in A \text{ and } 0 < |x - x_0| < \delta, \text{ then } |f(x) - l| < \epsilon.$$

ex: • Let $f(x) = \begin{cases} x^2, & x \in \mathbb{R} \setminus \{0\} \\ 1, & x=0 \end{cases}$



Notice that the domain of f is \mathbb{R} , and 0 is an accumulation point of \mathbb{R} . So, it makes sense to ask whether $\lim_{x \rightarrow 0} f(x)$ exists, and, if yes, what it is equal to.

We guess it should exist and be 0 . Indeed:

We want to show that: $\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. :}$

$$\text{if } \underbrace{0 < |x - 0| < \delta}_{\text{i.e. } 0 < |x| < \delta}, \text{ then } \underbrace{|f(x) - 0| < \epsilon}_{\text{i.e. } |x^2| < \epsilon, \text{ i.e. } |x|^2 < \epsilon}.$$

(7)

Let ε_0 . If $|x| < \varepsilon_0^{1/2}$ then $|x|^2 < \varepsilon$.

In particular, if $0 < |x| < \varepsilon_0^{1/2}$, then $|f(x)| < \varepsilon$.

Since ε_0 was arbitrary,

$\lim_{x \rightarrow 0} f(x) = 0$ (the definition works for $\delta = \varepsilon_0^{1/2}$; or for any $\delta \in (0, \varepsilon_0^{1/2})$, actually).

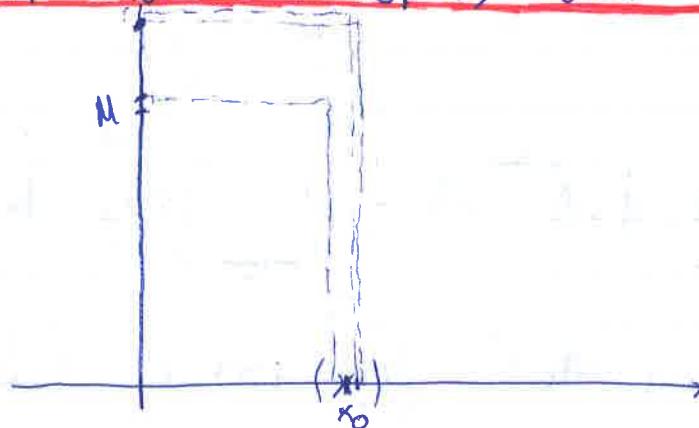


Notice that the value of f at 0 didn't matter at all; even if f was not defined at 0, it still wouldn't matter.

→ Def: Let $f: A \xrightarrow{\text{cp}} \mathbb{R}$, and x_0 an accumulation point of A .

Then, $\lim_{x \rightarrow x_0} f(x) = +\infty$ if: $\forall N > 0, \exists \delta > 0$ s.t.

if $x \in A$ and $0 < |x - x_0| < \delta$, then $f(x) > N$



(8)

And:

$$\lim_{x \rightarrow x_0} f(x) = -\infty \text{ if } \forall M > 0, \exists \delta > 0 \text{ s.t. :}$$

$$\text{if } x \in A \text{ and } 0 < |x - x_0| < \delta, \text{ then } f(x) < -M.$$

To define limits of a function as $x \rightarrow +\infty$ or $x \rightarrow -\infty$, we see $+\infty$ and $-\infty$ as accumulation points of the domain:

→ Def: Let $A \subseteq \mathbb{R}$. We say that $+\infty$ is an accumulation point of A if: $\forall M > 0$, there exists $x \in A$ with $x > M$.

ex: $A = [2, +\infty)$, $A = \mathbb{N}$, $A = \mathbb{Z}$, $A = \mathbb{Q}$.

x in the
"punctured"
neighbourhood
 $(M, +\infty)$ of $+\infty$.

Similarly, we say that $-\infty$ is an accumulation point of A if: $\forall M > 0$, there exists $x \in A$ with $x < -M$.

x in the "punctured"
neighbourhood $(-\infty, M)$
of $-\infty$.

→ Def: Let $f: A \xrightarrow{\subset} \mathbb{R}$, and let $+\infty$ be an accumulation point of A .

We say that $\lim_{x \rightarrow +\infty} f(x) = l$ (where $l \in \mathbb{R} \cup \{+\infty, -\infty\}$)

(3)

if, for any neighbourhood N_l of l , there exists
 a "punctured" neighbourhood $(N, +\infty)$ of $+\infty$, s.t.

$f((N, +\infty) \cap A) \subseteq N_l$. According to whether $l \in \mathbb{R}$, $l = +\infty$
 or $l = -\infty$, this becomes:

- $\lim_{x \rightarrow +\infty} f(x) = l \in \mathbb{R}$ if : $\forall \epsilon > 0 \ \exists N > 0$ s.t.
 if $x \in A, x > N$, then $|f(x) - l| < \epsilon$.
- $\lim_{x \rightarrow +\infty} f(x) = +\infty$ if : $\forall M_1 > 0 \ \exists M_2 > 0$ s.t.
 if $x \in A, x > M_2$, then $f(x) > M_1$.
- $\lim_{x \rightarrow +\infty} f(x) = -\infty$ if : $\forall M_1 > 0 \ \exists M_2 > 0$ s.t.
 if $x \in A, x < -M_2$, then $f(x) > M_1$.

We similarly define $\lim_{x \rightarrow -\infty} f(x) = l$, for $l \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Lecture 16:

(1)

Characterisation of accumulation points:

→ Theorem: Let $\emptyset \neq A \subseteq \mathbb{R}$, and $x_0 \in \mathbb{R} \cup \{\pm\infty, -\infty\}$.
 The following are equivalent:

- (i) x_0 is an accumulation point of A .
- (ii) Every neighbourhood of x_0 contains infinitely many elements of A , different from x_0 .
- (iii) There exists a sequence $(x_n)_{n \in \mathbb{N}}$ in A , all of whose terms are pairwise distinct (i.e. $x_{n_1} \neq x_{n_2} \forall n_1, n_2 \in \mathbb{N}$), with such that $\frac{x_n \neq x_0}{x_n} \rightarrow x_0$.

Proof: We will see here the proof for when $x_0 \in \mathbb{R}$.
 The proof for $x_0 = +\infty$ and $x_0 = -\infty$ is left as an exercise.

So: we have $x_0 \in \mathbb{R}$.

(i) \rightarrow (ii): Let $\delta > 0$. Suppose that the neighbourhood $(x_0 - \delta, x_0 + \delta)$ of x_0 contains finitely many elements $x_1, x_2, \dots, x_N \in A$.

(2)

Idea:

One of these is closest to x_0 , so closer to x_0 .
 there are no points of A , contradiction, as x_0 is an accumulation point of A .

$$\text{Let } \delta_0 := \min\{|x_1 - x_0|, |x_2 - x_0|, \dots, |x_N - x_0|\}.$$

Since x_0 is an accumulation point of A ,

there exists $y \in A$ (with $y \neq x_0$) with $|y - x_0| < \delta_0 = \min\{|x - x_0| : x \in A\}$,

contradiction. So, $(x_0 - \delta_0, x_0 + \delta_0)$ contains infinitely many elements of A .

Since $\delta_0 > 0$ was arbitrary, (ii) holds.

(ii) \rightarrow (iii): Since (ii) holds:

for $\delta = 1$: there exists $x_1 \in (A \cap (x_0 - 1, x_0 + 1)) \setminus \{x_0\}$,
 i.e. there exists $x_1 \in A$, $x_1 \neq x_0$, with $|x_1 - x_0| < 1$.

for $\delta = \frac{1}{2}$: there exists $x_2 \in (A \cap (x_0 - \frac{1}{2}, x_0 + \frac{1}{2})) \setminus \{x_0, x_1\}$,
 i.e. there exists $x_2 \in A$, $x_2 \neq x_0$, $x_2 \neq x_1$, with $|x_2 - x_0| < \frac{1}{2}$.

for $\delta = \frac{1}{n}$: there exists $x_n \in (A \cap (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})) \setminus \{x_0, x_1, \dots, x_{n-1}\}$,
 i.e. there exists $x_n \in A$, $x_n \neq x_0, x_1, \dots, x_{n-1}$,
 with $|x_n - x_0| < \frac{1}{n}$.

(3)

We have thus constructed a sequence $(x_n)_{n \in \mathbb{N}}$,

with

$$x_0 - \frac{1}{n} < x_n < x_0 + \frac{1}{n} \quad \forall n \in \mathbb{N} \Rightarrow x_n \rightarrow x_0,$$

$\downarrow n \rightarrow \infty \qquad \downarrow n \rightarrow \infty$

$x_0 \qquad \qquad x_0$

$$x_n \in A \setminus \{x_0\} \quad \forall n \in \mathbb{N},$$

$$\text{and } x_{n_1} \neq x_{n_2} \quad \forall n_1 \neq n_2$$

as required.

(iii) \Rightarrow (i) Let $\delta > 0$. We know that there exists

a sequence $(x_n)_{n \in \mathbb{N}}$, with terms $x_n \neq x_0 \quad \forall n \in \mathbb{N}$,
 s.t. $x_n \rightarrow x_0$. So:

$\exists n_0 \in \mathbb{N}$ s.t. $|x_{n_0} - x_0| < \delta$, for the fixed δ
 above. Of course, $x_{n_0} \neq x_0$.

Thus, $x_{n_0} \in A \cap ((x_0 - \delta, x_0 + \delta) \setminus \{x_0\})$,

so $A \cap ((x_0 - \delta, x_0 + \delta) \setminus \{x_0\}) \neq \emptyset$.

Since δ was arbitrary, x_0 is an accumulation point of A .

(4)

→ Characterisation of limits via limits of sequences :

→ Theorem: Let $f: A \xrightarrow{\text{CR}} \mathbb{R}$, and $x_0 \in \mathbb{R} \cup \{+\infty, -\infty\}$ an accumulation point of A .

The following are equivalent:

- (i) $\lim_{x \rightarrow x_0} f(x) = l$.
- (ii) For any sequence $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R} with $x_n \neq x_0$ for all $n \in \mathbb{N}$, such that $x_n \rightarrow x_0$, we have that $f(x_n) \rightarrow l$.

Proof: Here is the proof for $x_0, l \in \mathbb{R}$. The rest of the cases follow similarly, and are left as exercises.

(i) \Rightarrow (ii): Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R} , with $x_n \neq x_0$ $\forall n \in \mathbb{N}$, and $x_n \rightarrow x_0$. We will show that $f(x_n) \rightarrow l$.

Indeed, let $\epsilon > 0$. Since $\lim_{x \rightarrow x_0} f(x) = l$, $\exists \delta > 0$ s.t. $\begin{cases} \text{if } 0 < |x - x_0| < \delta, \text{ then } \\ |f(x) - l| < \epsilon. \end{cases}$

(5)

Since $x_n \rightarrow x_0$, we have that, for this δ ,

$\exists n_0 \in \mathbb{N}$ s.t. : $\forall n \geq n_0, |x_n - x_0| < \delta$.



Thus, by $\textcircled{*}$, $|f(x_n) - l| < \varepsilon, \forall n \geq n_0$.

Since $\varepsilon > 0$ was arbitrary, $f(x_n) \xrightarrow[n \rightarrow \infty]{} l$.

$\boxed{\text{(ii)} \rightarrow \text{(i)}} :$ Suppose that $\lim_{x \rightarrow x_0} f(x)$ is not l . Then, there

exists some $\varepsilon > 0$, such that : $\forall \delta > 0$,

there exists $x_\delta \in A$, with $0 < |x_\delta - x_0| < \delta$,
s.t. $|f(x_\delta) - l| \geq \varepsilon$.

i.e., $x_\delta \neq x_0$

Idea: If it is not true that $\forall \varepsilon > 0$ there exists some good $\delta > 0$,
then it must be that, for some $\varepsilon > 0$, all $\delta > 0$ are bad.

⑥

Thus:

- for $\delta=1$, $\exists x_1 \in A$, $x_1 \neq x_0$, with $x_0-1 < x_1 < x_0+1$,
and $|f(x_1) - l| \geq \epsilon$
- for $\delta=\frac{1}{2}$, $\exists x_2 \in A$, $x_2 \neq x_0$, with $x_0-\frac{1}{2} < x_2 < x_0+\frac{1}{2}$,
and $|f(x_2) - l| \geq \epsilon$.
- for $\delta=\frac{1}{n}$, $\exists x_n \in A$, $x_n \neq x_0$, with $x_0-\frac{1}{n} < x_n < x_0+\frac{1}{n}$,
and $|f(x_n) - l| \geq \epsilon$

We have thus constructed a sequence $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R} ,

with $x_n \in A \quad \forall n \in \mathbb{N}$,

$x_n \neq x_0 \quad \forall n \in \mathbb{N}$,

and $x_0 - \frac{1}{n} < x_n < x_0 + \frac{1}{n}$, so $x_n \rightarrow x_0$.

$$\begin{array}{ccc} \downarrow & & \downarrow \\ x_n & & x_0 \end{array}$$

$$\left. \begin{array}{c} (ii) \\ \xrightarrow{\hspace{1cm}} f(x_n) \rightarrow f(x_0) \end{array} \right\}$$

However, we also have that $|f(x_n) - l| \geq \epsilon \quad (\forall n \in \mathbb{N})$

so $f(x_n) \not\rightarrow l$, contradiction.

So, our initial assumption was wrong,

$$\text{so } \lim_{x \rightarrow x_0} f(x) = l.$$

(7)

An immediate Corollary is the following:

Corollary:

Let $f: A \xrightarrow{\text{CR}} \mathbb{R}$, $g: A \xrightarrow{\text{CR}} \mathbb{R}$,

and x_0 an accumulation point of A
(in $\mathbb{R} \cup \{\infty, -\infty\}$).

Suppose that $\lim_{x \rightarrow x_0} f(x) = l_1 \in \mathbb{R}$

and $\lim_{x \rightarrow x_0} g(x) = l_2 \in \mathbb{R}$.

Then:

- $\lim_{x \rightarrow x_0} (f+g)(x) = l_1 + l_2$
- $\lim_{x \rightarrow x_0} (f \cdot g)(x) = l_1 \cdot l_2$
- $\lim_{x \rightarrow x_0} \left(\frac{f}{g}\right)(x) = \frac{l_1}{l_2}$, if $l_2 \neq 0$.

Proof: A very simple exercise.

Continuity:

Let $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function, and $x_0 \in A$.

it doesn't

have to be an accumulation point of A !

We want to explain what we will mean by

" f is continuous at x_0 ". We will mean that:

"When x is very close to x_0 (in A), then $f(x)$ will be very close to $f(x_0)$."

More precisely:

"No matter how small a neighbourhood $(f(x_0)-\epsilon, f(x_0)+\epsilon)$

I consider, I can still find some (potentially tiny) neighbourhood $(x_0-\delta, x_0+\delta)$ of x_0 , such that

all points of the domain A of f in that neighbourhood are sent inside $(f(x_0)-\epsilon, f(x_0)+\epsilon)$ via f ".

notice that $f(x_0) \in (f(x_0)-\epsilon, f(x_0)+\epsilon)$ always.

(9)

I.e., "no matter how close to $f(x_0)$ I look at, when I look at x close enough to x_0 , $f(x)$ will be at most that close to $f(x_0)$."

Mainly, the only difference from saying that

$\lim_{x \rightarrow x_0} f(x) = f(x_0)$ is that x_0 is not necessarily an accumulation point of A .

→ Defi Let $f: A \xrightarrow{\text{CR}} \mathbb{R}$, and $x_0 \in A$

We say that f is continuous at x_0 if:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.: $f\left(A \cap \underbrace{(x_0 - \delta, x_0 + \delta)}_{\substack{\text{not} \\ \text{punctured at} \\ x_0 \text{ this time!}}}\right) \subseteq (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

[Δ perhaps depending on ε, x_0 !]

i.e. if $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon, x_0)$ s.t.:

if $x \in A$ and $|x - x_0| < \delta$, then: $|f(x) - f(x_0)| < \varepsilon$.

(10)

Now, you could argue that this is exactly the same as saying that $\lim_{x \rightarrow x_0} f(x) = f(x_0)$; since x_0 is sent to $f(x_0) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$ anyway, the definition just requires me to check whether $f(A \cap \underbrace{(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}}_{\text{punctured}}) \subseteq (f(x_0) - \epsilon, f(x_0) + \epsilon)$

Sure; but what if there are no points at all in $A \cap ((x_0 - \delta, x_0 + \delta) \setminus \{x_0\})$, i.e. what if x_0 is an isolated point of A ? We want to still be able to discuss continuity of f at isolated points of A ; and that is the only reason why we don't just say that f is continuous at x_0 if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

Notice that, when x_0 is an isolated point of A , then there exists a neighbourhood N_{x_0} of x_0 that contains no other points of A other than x_0 ;

(11)

so, for any neighbourhood $(f(x_0)-\varepsilon, f(x_0)+\varepsilon)$ of $f(x_0)$,
we have that $\underbrace{f(A \cap N_{x_0})}_{\text{if } f(x_0)} \subseteq (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$!

So, f is continuous at x_0 . We thus have the following:

Let $f: A \rightarrow \mathbb{R}$, and $x_0 \in A$.

- If x_0 is an isolated point of A , then f is continuous at x_0 .
- If x_0 is an accumulation point of A , then f is continuous at x_0 if and only if

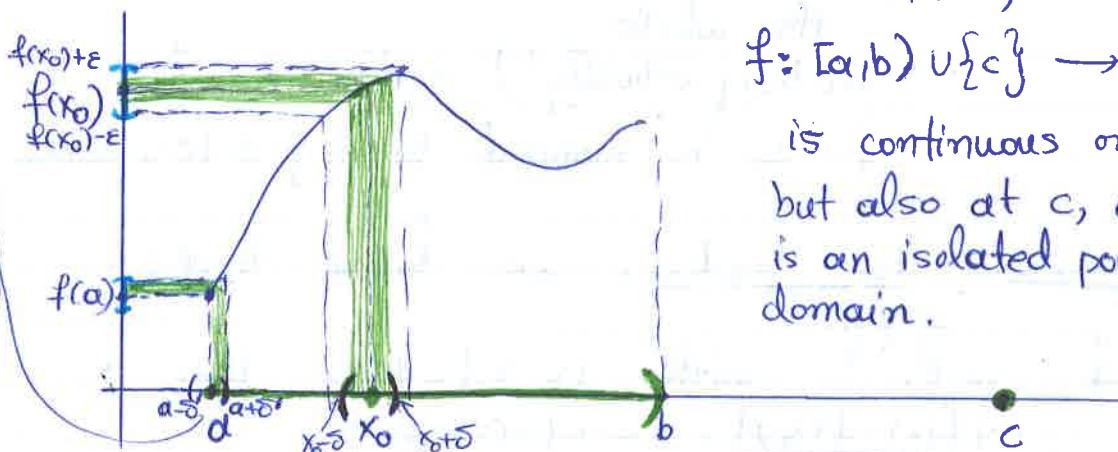
$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

for instance, this

$$f: [a, b] \cup \{c\} \rightarrow \mathbb{R}$$

is continuous on $[a, b]$, but also at c , as c is an isolated point of the domain.

See how every x in A in $(a-\delta, a+\delta')$ is sent into $(f(a)-\varepsilon, f(a)+\varepsilon)$ via f !



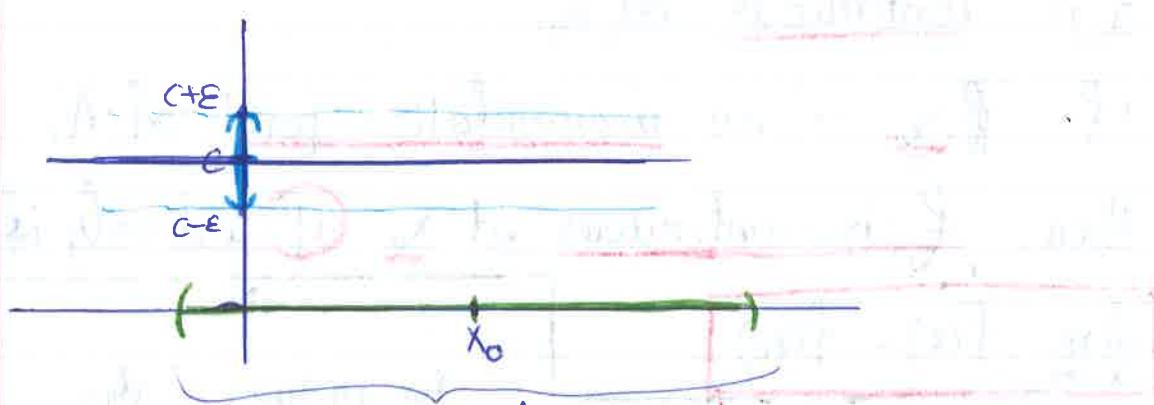
(12)

A. Morally, f is continuous at x_0 if we can draw the graph of f infinitesimally close to $(x_0, f(x_0))$ without lifting our pencil from the paper.

→ Examples:

(1) $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = c \forall x \in \mathbb{R}$: f is continuous at every $x_0 \in \mathbb{R}$.

Indeed, let $x_0 \in \mathbb{R}$.



this whole
(arbitrary actually!) neighbourhood
of x_0 is mapped to $\{c\} \subseteq (c-\epsilon, c+\epsilon)$.

Let $\epsilon > 0$. For $\delta = 100$, we have that:

If $x \in \mathbb{R}$ ^{domain of f} with $|x - x_0| < 100$, then
 $|f(x) - f(x_0)| = |c - c| = 0 \leq \epsilon$.

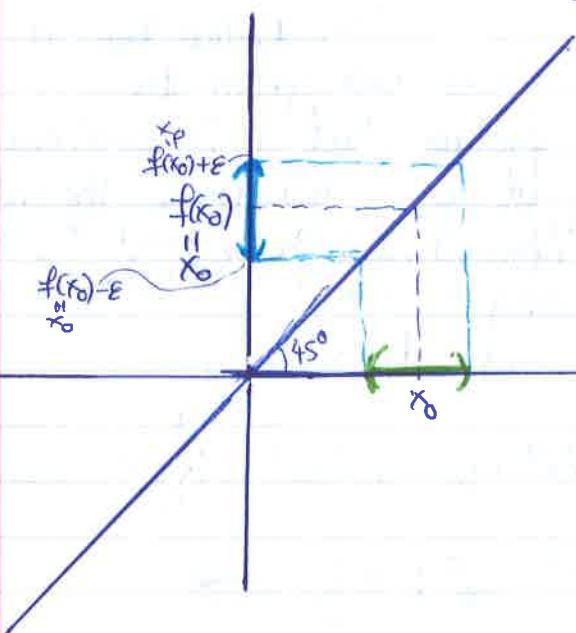
(13)

Since $\varepsilon > 0$ was arbitrary, f is continuous at x_0 .

Since $x_0 \in \mathbb{R}$ was arbitrary, f is continuous on the whole of \mathbb{R} .

(2) $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x \quad \forall x \in \mathbb{R}$: f is continuous at every $x_0 \in \mathbb{R}$.

Indeed, let $x_0 \in \mathbb{R}$. Let $\varepsilon > 0$.



Let $\delta = \varepsilon$ (> 0).

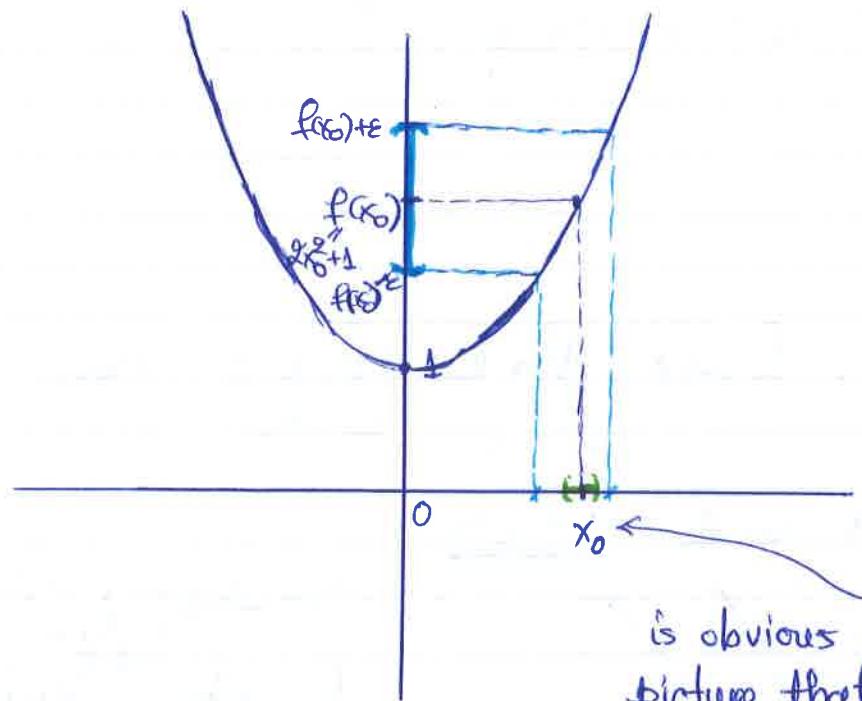
If $x \in \mathbb{R}$ and $|x - x_0| < \delta$,

then $|f(x) - f(x_0)| = |x - x_0| \leq \varepsilon$.

Since $\varepsilon > 0$ was arbitrary,
 f is continuous at x_0 .

Since $x_0 \in \mathbb{R}$ was arbitrary,
 f is continuous on the whole of \mathbb{R} .

$$(3) \quad f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = 2x^2 + 1, \quad \forall x \in \mathbb{R}.$$



Let $x_0 \in \mathbb{R}$. Let $\epsilon > 0$.

Notice how it is obvious from the picture that f is continuous at x_0 . There certainly exists a whole interval around x_0 , mapped inside $(f(x_0)-\epsilon, f(x_0)+\epsilon)$ via f .

We want to find $\delta > 0$, s.t.: if $|x-x_0| < \delta$,

$$\text{then } |f(x) - f(x_0)| < \epsilon \Leftrightarrow |(2x^2 + 1) - (2x_0^2 + 1)| < \epsilon \Leftrightarrow$$

$$\Leftrightarrow |2x^2 - 2x_0^2| < \epsilon \Leftrightarrow |x^2 - x_0^2| < \frac{\epsilon}{2} \Leftrightarrow$$

$$\Leftrightarrow \underbrace{|x-x_0| \cdot |x+x_0|}_{\text{we write}} < \frac{\epsilon}{2}$$

$$|x-x_0| = |x-x_0| \cdot |x+x_0|$$

so that $|x-x_0|$ will show up (the quantity we understand "well")

\hookrightarrow a matter of speech, as we don't know what δ to use yet!

(15)

Idea: if we could bound $|x+x_0|$ from above

by sth independent of ε, x (but not necessarily independent of x_0 !),

then we could control $|x-x_0| \cdot |x+x_0|$.

Notice that $|x+x_0| = |\underbrace{x-x_0}_{\text{(*)}} + 2x_0| \leq |x-x_0| + 2|x_0|$.

So, when $|\underbrace{x-x_0}_{\text{(*)}}| < 1$, then $|x+x_0| < \underbrace{1+2|x_0|}_{\text{independent of } x, \varepsilon}$,

remember, only care about what happens when x is close to x_0

thus $|x-x_0| \cdot |x+x_0| < |x-x_0| \cdot (1+2|x_0|)$, when $|\underbrace{x-x_0}_{\text{(*)}}| < 1$.

So, when at the same time

$$|\underbrace{x-x_0}_{\text{(**)}}| < \frac{\varepsilon}{1+2|x_0|} \quad (\text{(**)})$$

then $|\underbrace{x-x_0}_{\text{(*)}} \cdot |x+x_0| < \varepsilon$, thus $|f(x) - f(x_0)| < \varepsilon$.

Thus, I just require that (*) and (**) simultaneously hold, i.e. I get $\delta := \min\left\{1, \frac{\varepsilon}{1+2|x_0|}\right\} (>0)$.

for this δ , we have shown that:

if $x \in \mathbb{R}$, $|x-x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, f is continuous at x_0 .

Since $x_0 \in \mathbb{R}$ was arbitrary, f is continuous on the whole of \mathbb{R} .

(16)

2nd way (not with ϵ - δ definition):

Let $x_0 \in \mathbb{R}$. Since x_0 is an accumulation point of the domain \mathbb{R} of f , we have that:

f is continuous at x_0 if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$,

thus if :

for any sequence $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R} ,
↓
domain of f

with $x_n \neq x_0 \forall n \in \mathbb{N}$ and $x_n \rightarrow x_0$,

we have that $f(x_n) \rightarrow f(x_0)$.

So, it suffices to verify $\textcircled{*}$. Indeed, let $(x_n)_{n \in \mathbb{N}}$ be st. $x_n \neq x_0 \forall n \in \mathbb{N}$, with $x_n \rightarrow x_0$.

Then, $x_n^2 \xrightarrow{n \rightarrow \infty} x_0^2$, i.e. $f(x_n) \rightarrow f(x_0)$.

So, $\textcircled{*}$ holds, so f is continuous at x_0 .

Since $x_0 \in \mathbb{R}$ was arbitrary, f is continuous on \mathbb{R} .

See how much easier the 2nd way was; and, for continuity, we have an even nicer characterisation!

Lecture 17:

30 Oct 2016

→ Characterisation of continuity via limits of sequences:

→ Theorem: Let $f: A \xrightarrow{\text{CR}} \mathbb{R}$, and $x_0 \in A$. Then,

f continuous at x_0

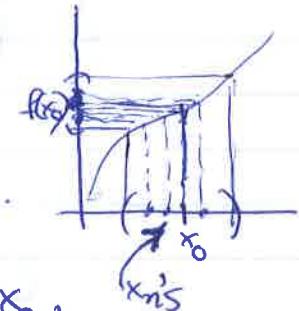


for every sequence $(x_n)_{n \in \mathbb{N}}$ in A , with $x_n \xrightarrow{} x_0$,
we have $f(x_n) \xrightarrow{} f(x_0)$.

we don't require
 $x_n \neq x_0$ then.

Proof: (\Rightarrow) Let $(x_n)_{n \in \mathbb{N}}$ in A , with $x_n \xrightarrow{} x_0$.

We want to show: $f(x_n) \xrightarrow{} f(x_0)$.



Let $\epsilon > 0$. Since f is continuous at x_0 ,

there exists some $\delta > 0$ s.t.:

whenever $x \in A$ and $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$. $\quad \textcircled{*}$

Since $x_n \xrightarrow{} x_0$, $\exists n_0 \in \mathbb{N}$ s.t. : $\forall n \geq n_0$, $|x_n - x_0| < \delta$.

By $\textcircled{*}$, $|f(x_n) - f(x_0)| < \epsilon$, $\forall n \geq n_0$.

So, $f(x_n) \xrightarrow{} f(x_0)$.

②

(\Leftarrow) Suppose that f is not continuous at $x_0 \in A$.

We will show that: There exists

a sequence $(x_n)_{n \in \mathbb{N}}$ in A , for which:

$$x_n \rightarrow x_0, \text{ but } f(x_n) \not\rightarrow f(x_0)$$

(which is a contradiction).

Indeed: Since f is not continuous at $x_0 \in A$,

there exists some neighbourhood $(f(x_0)-\varepsilon, f(x_0)+\varepsilon)$

of $f(x_0)$, st. $(x_0-\delta, x_0+\delta)$ is not sent inside
 $(f(x_0)-\varepsilon, f(x_0)+\varepsilon)$, for any $\delta > 0$.

i.e.: $\exists \varepsilon > 0$ st. :

$\forall \delta > 0$, there exists $x_\delta^A \in (x_0-\delta, x_0+\delta)$ st.
 $f(x_\delta) \notin (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$.

i.e., for this $\varepsilon > 0$:

$\forall \delta > 0$, $\exists x_\delta \in A$, with $|x_\delta - x_0| < \delta$,

s.t. $|f(x_\delta) - f(x_0)| \geq \varepsilon$.

(3)

We apply this for $\delta = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$;

we have that, $\forall n \in \mathbb{N}$, there exists $x_n \in A$, with $|x_n - x_0| < \frac{1}{n}$, such that $|f(x_n) - f(x_0)| \geq \varepsilon$.

Since $|x_n - x_0| < \frac{1}{n} \quad \forall n \in \mathbb{N}$

$$\Leftrightarrow -\frac{1}{n} + x_0 < x_n < \frac{1}{n} + x_0 \quad \forall n \in \mathbb{N},$$

$$\downarrow_{n \rightarrow \infty} \qquad \qquad \qquad \downarrow_{n \rightarrow \infty}$$

$$x_0 \qquad \qquad \qquad x_0$$

$$\text{we have: } x_n \xrightarrow[n \rightarrow \infty]{} 0.$$

However, $|f(x_n) - f(x_0)| \geq \varepsilon (> 0) \quad \forall n \in \mathbb{N}$,

$$\text{so } |f(x_n) - f(x_0)| \xrightarrow[n \rightarrow \infty]{} 0,$$

$$\text{i.e. } f(x_n) - f(x_0) \xrightarrow[n \rightarrow \infty]{} 0,$$

$$\text{i.e. } f(x_n) \xrightarrow[n \rightarrow \infty]{} f(x_0).$$

Examples:

(i) Notice that the previous example ($f(x) = 2x^2 + 1$) has an even easier solution now! :

Let $x_0 \in \mathbb{R}$. Let $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R} , with $x_n \xrightarrow[n \rightarrow \infty]{} x_0$.

$$\text{Then, } 2x_n^2 + 1 \xrightarrow[n \rightarrow \infty]{} 2x_0^2 + 1, \text{ i.e. } f(x_n) \rightarrow f(x_0).$$

(4)

Since $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R} was arbitrary, we have:

f continuous at x_0 . Since $x_0 \in \mathbb{R}$ was arbitrary, we have: f continuous on \mathbb{R} .

(ii)

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \quad \sim \text{known as} \\ \text{"the Dirichlet function".}$$

f is discontinuous (i.e. not continuous) at every $x_0 \in \mathbb{R}$. Indeed:

Let $x_0 \in \mathbb{R}$. We know that:

- there exists a sequence $(q_n)_{n \in \mathbb{N}}$ of rationals, with $q_n \rightarrow x_0$.
- there exists a sequence $(a_n)_{n \in \mathbb{N}}$ of irrationals, with $a_n \rightarrow x_0$.

Suppose that f is continuous at x_0 . Since

$(q_n)_{n \in \mathbb{N}}$ and $(a_n)_{n \in \mathbb{N}}$ are sequences in the domain of f (i.e., \mathbb{R}), we must have that

$$\underbrace{f(q_n)}_{\substack{\rightarrow \\ "1, n \in \mathbb{N}"}} \underset{n \rightarrow \infty}{\rightarrow} f(x_0) \quad \text{and} \quad \underbrace{f(a_n)}_{\substack{\rightarrow \\ "0, n \in \mathbb{N}"}} \underset{n \rightarrow \infty}{\rightarrow} f(x_0)$$

(5)

So, $f(x_0) = 1$ and $f(x_0) = 0$, contradiction.

So, f is discontinuous at x_0 . And since $x_0 \in \mathbb{R}$ was arbitrary, f is discontinuous everywhere on \mathbb{R} .

$$(iii) f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ x^3, & x \notin \mathbb{Q} \end{cases}$$

Suppose that f is continuous at $x_0 \in \mathbb{R}$.

Let $(q_n)_{n \in \mathbb{N}}$ in \mathbb{Q} with $q_n \xrightarrow[n \rightarrow \infty]{} x_0$,
and $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{R} \setminus \mathbb{Q}$ with $a_n \xrightarrow[n \rightarrow \infty]{} x_0$.

Due to our assumption that f is continuous at x_0 ,

we have: $\underbrace{f(q_n)}_{\substack{\parallel \\ q_n}} \xrightarrow[n \rightarrow \infty]{} f(x_0) \rightarrow f(x_0) = x_0,$

and $\underbrace{f(a_n)}_{\substack{\parallel \\ a_n^3}} \xrightarrow[n \rightarrow \infty]{} f(x_0) \xrightarrow[a_n \rightarrow x_0]{} f(x_0) = x_0^3.$

So, unless $x_0^3 = x_0 \Leftrightarrow x_0^3 - x_0 = 0 \Leftrightarrow x_0(x_0^2 - 1) = 0$
 $\Leftrightarrow x_0(x_0 - 1)(x_0 + 1) = 0$
 $\Leftrightarrow x_0 = 0 \text{ or } x_0 = 1 \text{ or } x_0 = -1,$

f cannot be continuous at x_0 .

(6)

f is actually continuous at $0, 1, -1$ (exercise).

→ Theorem: Let $f, g : A \rightarrow \mathbb{R}$ be continuous at $x_0 \in A$.

Then, $f+g$, $\lambda \cdot f$, $f \cdot g$ are continuous at x_0 . Also, if $g(x) \neq 0 \forall x \in A$, then $\frac{f}{g}$ is continuous at x_0 .

Proof: Let $(x_n)_{n \in \mathbb{N}}$ in A be s.t. $x_n \rightarrow x_0$.

Since f, g are continuous at x_0 , we have:

$$f(x_n) \rightarrow f(x_0) \text{ and } g(x_n) \rightarrow g(x_0)$$

So; $\underbrace{f(x_n) + g(x_n)}_{(f+g)(x_n)} \longrightarrow \underbrace{f(x_0) + g(x_0)}_{f+g(x_0)}$. Since $(x_n)_{n \in \mathbb{N}}$ in A was arbitrary, $f+g$ continuous at x_0 .

\bullet $\underbrace{\lambda \cdot f(x_n)}_{(\lambda f)(x_n)} \longrightarrow \underbrace{\lambda \cdot f(x_0)}_{(\lambda f)(x_0)}$. Since $(x_n)_{n \in \mathbb{N}}$ in A was arbitrary, λf cont. at x_0 .

The rest follow similarly. ■

(7)

→ Def: Let $f: A \rightarrow \mathbb{R}$. We say that f is continuous if it is continuous at every $x_0 \in A$.

→ Corollary: Let $f: A \rightarrow \mathbb{R}$, $g: A \rightarrow \mathbb{R}$, both continuous. Then, $f+g$, $f \cdot g$ and $\overset{\mathbb{R}}{f} \cdot f$ are continuous. Also, if $g(x) \neq 0$ for $x \in A$, then $\frac{f}{g}$ is continuous.

→ Examples : • $x^2, x^3, \dots, x^n, \dots$ are continuous functions on \mathbb{R} .
• Every polynomial $p: \mathbb{R} \rightarrow \mathbb{R}$ is continuous. (ex.: $2x^2 + 1$).

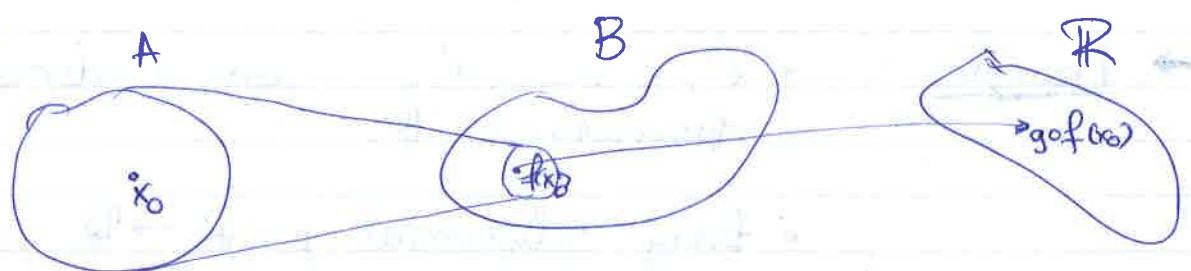
(A polynomial $p: \mathbb{R} \rightarrow \mathbb{R}$ is any function of the form $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m$, where $m \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \dots, a_m \in \mathbb{R}$).



→ Theorem: Let $f: A \xrightarrow{\text{CR}} \mathbb{R}$, $g: B \xrightarrow{\text{CR}} \mathbb{R}$, with $f(A) \subseteq B$.

Suppose that f is continuous at $x_0 \in A$,
and g is continuous at $f(x_0) \in B$.
Then, gof is continuous at x_0 .

Proof:



Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in A , with $x_n \rightarrow x_0$.
→ the domain of f .

Since f is continuous at x_0 , $f(x_n) \rightarrow f(x_0)$.

$(f(x_n))_{n \in \mathbb{N}}$ is a sequence in B , with $f(x_n) \rightarrow f(x_0)$.
→ the domain of g .

Since g is continuous at $f(x_0)$, we have:

$$g(f(x_n)) \rightarrow g(f(x_0)),$$

i.e. $gof(x_n) \rightarrow gof(x_0)$. Since $(x_n)_{n \in \mathbb{N}}$ in A was arbitrary, gof is continuous at x_0 . ■

①

Local properties of continuous functions:

① Continuity is a local property. i.e.:

→ Thm: Let $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function, and

let $\tilde{f}: A \cap (x_0 - \delta, x_0 + \delta) \rightarrow \mathbb{R}$, with $\tilde{f}(x) = f(x)$

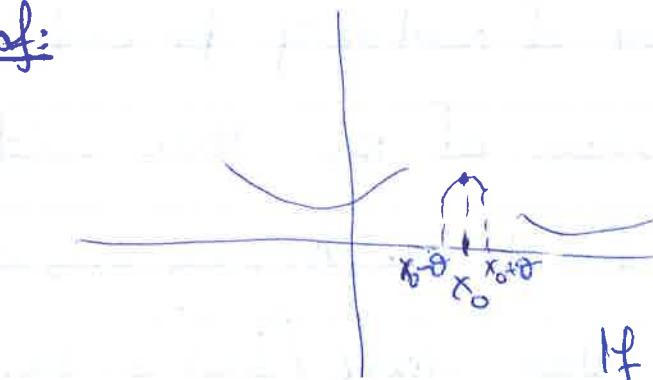
↓
the restriction
of f on $(x_0 - \delta, x_0 + \delta)$

$\forall x \in A \cap (x_0 - \delta, x_0 + \delta)$,

for some $\delta > 0$.

If \tilde{f} is continuous at x_0 , then f is continuous at x_0 .

Proof:



Let $\epsilon > 0$.

Since \tilde{f} is continuous at x_0 , there exists $\delta' > 0$ s.t. :

If $x \in A \cap (x_0 - \delta, x_0 + \delta)$, and $|x - x_0| < \delta'$, then $|f(x) - f(x_0)| < \epsilon$.

In particular: if $\delta := \min\{\delta, \delta'\}$, and $x \in A$ with $|x - x_0| < \delta$,

then $x \in A$, $x \in (x_0 - \delta, x_0 + \delta)$ and $|x - x_0| < \delta'$, so

$|f(x) - f(x_0)| < \epsilon$. So, f continuous at x_0 . ■

2

2 If f is continuous at x_0 , then f is bounded in a neighbourhood of x_0 . i.e.:

→ Thm: Let $f: A \xrightarrow{\text{CR}} \mathbb{R}$ be continuous at $x_0 \in A$.

Then, there exist $\delta > 0$ and $M > 0$ such that:

$$|f(x)| \leq M, \text{ for all } x \in A \cap (x_0 - \delta, x_0 + \delta).$$

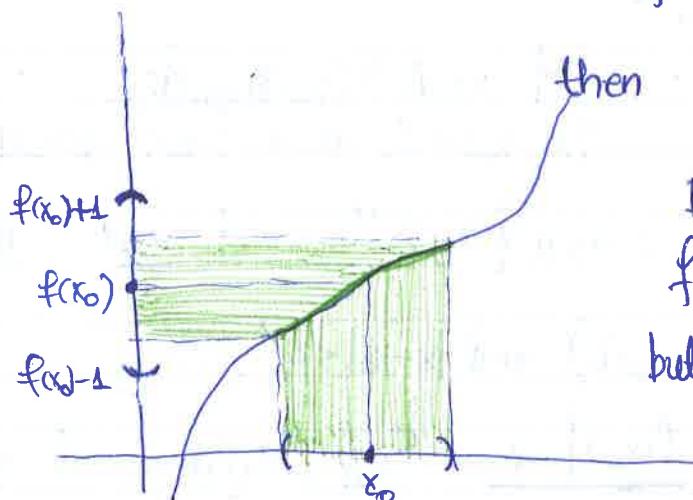
Proof:

Idea: We know that, since f continuous at x_0 , when x is close enough to x_0 , $f(x)$ will be close to $f(x_0)$; so $f(x)$ cannot take values too far from $f(x_0)$.

We apply the definition of continuity for $\epsilon = 1$:

Since f is continuous at x_0 , there exists

some $\delta > 0$ s.t. : if $x \in A \cap (x_0 - \delta, x_0 + \delta)$,



then $f(x) \in (f(x_0) - 1, f(x_0) + 1)$

It is proved already that
 f is bounded on $(x_0 - \delta, x_0 + \delta)$;

but let's bring our answer
in the form stated by
the theorem:

(3)

for $x \in A \cap (x_0 - \delta, x_0 + \delta)$, we have:

$$|f(x) - f(x_0)| < 1 \Rightarrow |f(x)| - |f(x_0)| (\leq |f(x) - f(x_0)|) < 1$$

$$\rightarrow |f(x)| < 1 + |f(x_0)|$$

$\underbrace{1 + |f(x_0)|}_{M > 0}$

③ Local preservation of sign around points of continuity:

→ Thm: Let $f: A \rightarrow \mathbb{R}$ be continuous at $x_0 \in A$, with $f(x_0) \neq 0$

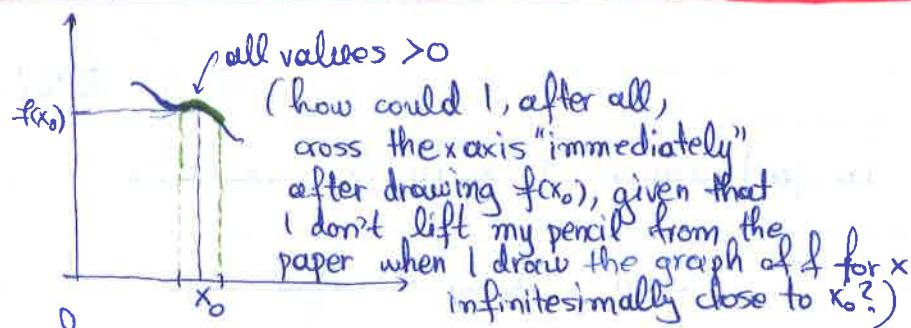
Then, there exists some neighbourhood of x_0 , such that the values of f on that neighbourhood have the same sign as $f(x_0)$.

I.e: - If $f(x_0) > 0$, then there exists $\delta > 0$, such that

$f(x) > 0$, for all $x \in A \cap (x_0 - \delta, x_0 + \delta)$.

- If $f(x_0) < 0$, then there exists $\delta > 0$, such that

$f(x) < 0$, for all $x \in A \cap (x_0 - \delta, x_0 + \delta)$.



4

Proof: $\left(\begin{array}{l} \text{Idea: If } f(x_0) > 0, \text{ then some neighbourhood} \\ \text{of } f(x_0) \text{ contains only positive numbers. So,} \\ \text{the area of } x_0 \text{ mapped in there can only be mapped} \\ \text{to positive values.} \end{array} \right)$

- Let $f(x_0) > 0$. f is continuous at x_0 . So,

for $\epsilon = \frac{f(x_0)}{2}$, there exists $\delta > 0$, such that if $x \in A \cap (x_0 - \delta, x_0 + \delta)$, then $f(x) \in \left(f(x_0) - \frac{f(x_0)}{2}, f(x_0) + \frac{f(x_0)}{2} \right)$

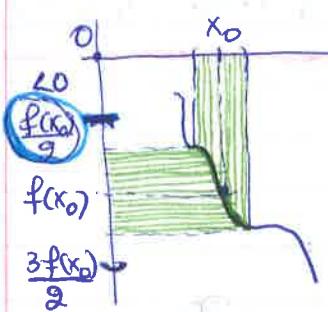
In particular, if $x \in A \cap (x_0 - \delta, x_0 + \delta)$,
then $f(x) > \frac{f(x_0)}{2} > 0$.

- Let $f(x_0) < 0$. f is continuous at x_0 . So,

there exists $\delta > 0$, such that:

if $x \in A \cap (x_0 - \delta, x_0 + \delta)$, then $f(x) \in \left(f(x_0) - \frac{|f(x_0)|}{2}, f(x_0) + \frac{|f(x_0)|}{2} \right)$

$$f(x_0) - \frac{|f(x_0)|}{2} = \frac{f(x_0)}{2} < 0$$



In particular, if $x \in A \cap (x_0 - \delta, x_0 + \delta)$,

then $f(x) < \frac{f(x_0)}{2} < 0$.

→ Important properties of functions that are continuous on closed intervals:

We will use the following tools to show the properties:

- Completeness of ordered field \mathbb{R} .
- Bolzano-Weierstrass theorem.
- Nested intervals
- Characterisation of continuity via limits of sequences.

→ Theorem 1: Every continuous function

$f: [a, b] \rightarrow \mathbb{R}$ is bounded.



Proof: We will show that f is bounded above and below.

(6)

- Suppose that f is not bounded above.

Then, for all $M > 0$, there exists $x_M \in [a, b]$,

such that $f(x_M) > M$. In particular:

for $M=1$, there exists $x_1 \in [a, b]$, s.t. $f(x_1) > 1$

for $M=2$, there exists $x_2 \in [a, b]$, s.t. $f(x_2) > 2$

⋮

for all $n \in \mathbb{N}$, there exists $x_n \in [a, b]$, s.t. $f(x_n) > n$.

Thus, $f(x_n) \rightarrow +\infty$.

Now: - $(x_n)_{n \in \mathbb{N}}$ is bounded, as $x_n \in [a, b] \forall n \in \mathbb{N}$.

So, by the Bolzano-Weierstrass theorem,
it has a subsequence $(x_{k_n})_{n \in \mathbb{N}}$

such that $\underset{n \rightarrow \infty}{\lim} x_{k_n} = x_0$, for some $x_0 \in \mathbb{R}$.

- $x_0 \in [a, b]$: Indeed, $a \leq x_{k_n} \leq b \forall n \in \mathbb{N}$,

so $\underset{n \rightarrow \infty}{\lim} x_{k_n} \in [a, b]$.

Since f is continuous at x_0 and $\underset{n \rightarrow \infty}{\lim} x_{k_n} = x_0$ because $x_0 \in [a, b]$
 $\underset{n \rightarrow \infty}{\lim} x_{k_n} = x_0$ in domain of f

it follows that $\underset{n \rightarrow \infty}{\lim} f(x_{k_n}) = f(x_0)$.

(7)

However, $f(x_{k_n}) \rightarrow +\infty$, as $(f(x_{k_n}))_{n \in \mathbb{N}}$ is a subsequence of $(f(x_n))_{n \in \mathbb{N}}$, whose limit is $+\infty$.

So, $f(x_0) = +\infty$, contradiction (as $f(x_0) \in \mathbb{R}$).

Thus, f is bounded above. Similarly, we show that f is bounded below. So, f is bounded.



What would go wrong in the above proof if we had $f: (\underbrace{a, b)}_{\text{open interval}} \rightarrow \mathbb{R}$, rather than $f: [a, b] \rightarrow \mathbb{R}$?

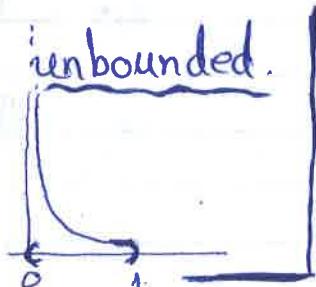
The only difference would be that the third " \rightsquigarrow " wouldn't work any more: it could be

that $x_{k_n} \xrightarrow[n \rightarrow \infty]{} a^+$, which is not a point of the domain of f now, in which case we cannot deduce that $(f(x_{k_n}))_{n \in \mathbb{N}}$

(and thus we get no contradiction: $f(x_{k_n}) \rightarrow +\infty$ and that's all.)

crucial point: in fact, there are examples of continuous functions $f: (a, b) \rightarrow \mathbb{R}$ that are unbounded.

- ex: $f: (0, 1) \rightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$ $\forall x \in (0, 1)$.



①

Lecture 19

→ **Theorem 2:** Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous.

Then, f has a minimal and a maximal value.

i.e., there exist $y, y' \in [a, b]$, such that:

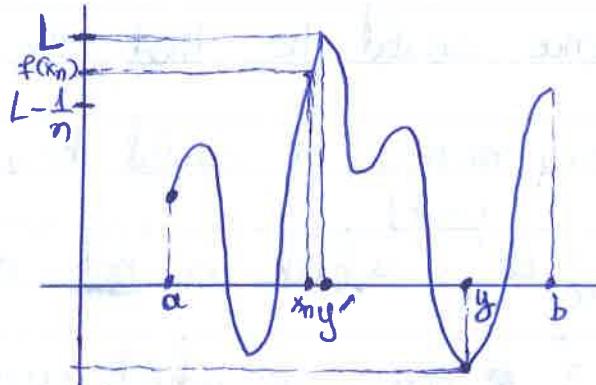
$$f(y) = \min \{f(x) : x \in [a, b]\} \quad \text{and}$$

$$f(y') = \max \{f(x) : x \in [a, b]\}$$

(in other words, $f(y) \leq f(x) \leq f(y')$, $\forall x \in [a, b]$).

⚠ This Theorem implies that f is bounded (Theorem 1).]

Proof:



Let $A := f([a, b]) = \{f(x) : x \in [a, b]\}$.

We will show that f has a maximal value,

i.e. that $\exists y' \in [a, b]$ st. $f(x) \leq f(y')$, $\forall x \in [a, b]$

(and similarly it can be shown that f has a

i.e.;
sup(f([a, b]))
inf(f([a, b]))
both
belong to
 $f([a, b])$

(2)

minimal value).

- We start by showing that A has a least upper bound L :

- $A \neq \emptyset$: indeed, $f(a) \in A$.

- A is bounded from above: by Theorem 1.

So, since $A \subseteq \mathbb{R}$ and \mathbb{R} is a complete ordered field, A has a least upper bound L .

- We will now show that $\exists y' \in [\alpha, b]$ such that $f(y') = L$

(i.e., that L is a maximum of A : $\sup A \in A$)

Idea: We will construct a convergent sequence in $[\alpha, b]$, whose image converges to L . By continuity, L must be the image of the limit of our convergent sequence!

For all $n \in \mathbb{N}$, $L - \frac{1}{n}$ is not an upper bound of $A = f([\alpha, b])$, so there exists some element of A larger than $L - \frac{1}{n}$; i.e., there exists $x_n \in [\alpha, b]$ s.t.

$$\boxed{L - \frac{1}{n} < f(x_n) \leq L}$$

$\downarrow n \rightarrow \infty$ $\downarrow n \rightarrow \infty$

L L

because L is an upper bound of $A = f([\alpha, b])$.

Thus: $f(x_n) \xrightarrow{n \rightarrow \infty} L$.

(3)

Now, since $x_n \in [a, b]$ for all $n \in \mathbb{N}$, we have that

$(x_n)_{n \in \mathbb{N}}$ is bounded $\xrightarrow{\text{Bolzano-Weierstrass}}$ $(x_n)_{n \in \mathbb{N}}$ has a convergent subsequence $(x_{k_n})_{n \in \mathbb{N}}$

Again since $a \leq x_{k_n} \leq b$ for all $n \in \mathbb{N}$,

we have $y' := \lim_{n \rightarrow \infty} x_{k_n} \in [a, b]$

the domain of f .

So, f is continuous at y' .

Thus, since $x_{k_n} \xrightarrow{n \rightarrow \infty} y'$, we have that

$$f(x_{k_n}) \xrightarrow{n \rightarrow \infty} f(y')$$

However, $f(x_{k_n}) \xrightarrow{n \rightarrow \infty} L$,

as $(f(x_{k_n}))_{n \in \mathbb{N}}$ is a subsequence of $(f(x_n))_{n \in \mathbb{N}}$,
and $f(x_n) \xrightarrow{n \rightarrow \infty} L$
(by its construction).

By uniqueness of limits, $f(y') = L$.

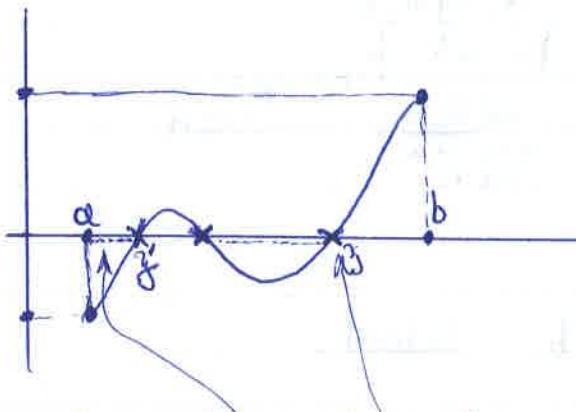
- Thus, $f(y')$ is the supremum of $f([a, b])$, which implies that $f(x) \leq f(y')$, for $x \in [a, b]$.

Theorem 3: Intermediate value theorem (Bolzano):

Let $f: [a, b] \rightarrow \mathbb{R}$ continuous,

with $f(a) < 0 < f(b)$. Then, there exists $\gamma \in (a, b)$ with $f(\gamma) = 0$

Proof:



Idea: We have shown that, for $x \in [a, b]$ very close to a , $f(x) < 0$ (sign preservation). As x increases though, at some point, $f(x)$ will stop being negative. The point where this change occurs will be a γ with $f(\gamma) = 0$.
 So, $\gamma := \sup \{x \in [a, b] : f(x) < 0\}$ for all $y \in [a, x]$ γ should work... To make things a little simpler, we notice that $\gamma := \sup \{x \in [a, b] : f(x) < 0\}$ should work too...
 (γ is the last place where the flip from - to + happens)
 (not the first; even though the first (γ') would work too.)

Let $A := \{x \in [a, b] : f(x) < 0\}$.

- $A \neq \emptyset$ (as $a \in A$, since $f(a) < 0$).
- A bounded from above (as $A \subseteq [a, b]$).
 So, since \mathbb{R} is a complete ordered field,

5

A has at least one upper bound $\sup A$

Clearly, $a \leq \gamma \leq b$

$a \in A$

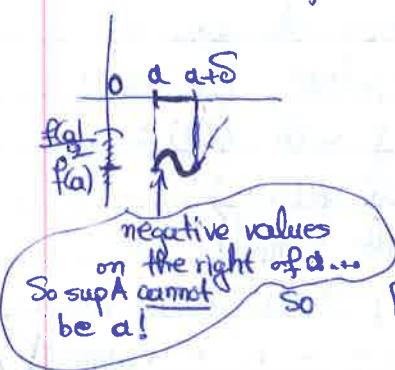
$\Rightarrow a \leq \gamma$

γ is an upper bound of A .

$A \subseteq [a, b]$,
 so b is an upper
 bound of A , so
 $\exists = \sup A \leq b$
 ↓ the least upper bound of A .
 ↑ an upper bound of A .

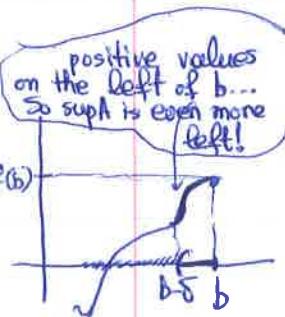
- Claim : $a \leq y \leq b$. Indeed :

- $f(a) < 0$, so there exists $\delta > 0$ s.t.



(due to the fact that f is continuous at a),
 +, thus $\sup_{\substack{\text{A} \\ \cap \\ \mathbb{R}}} A \geq a + \delta > a.$

- $f(b) > 0$, so there exists $\delta' > 0$ s.t.



$$f(x) \left(> \frac{f(b)}{2} \right) > 0, \text{ for all } x \in (b-\delta, b]$$

(as f is continuous at b),

$$A \subseteq [a, b - \delta'] \Rightarrow \sup_{\substack{\text{in } A \\ \text{if } \exists}} A \leq b - \delta' < b.$$

- Since $\hat{y} = \sup A$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in A,

with $x_n \xrightarrow[n \rightarrow \infty]{\quad} \hat{y}$ (exercise!).

Since f is continuous at \hat{y} , we have: $f(x_n) \xrightarrow[n \rightarrow \infty]{\quad} f(\hat{y})$.

- But $x_n \in A \ \forall n \in \mathbb{N} \Rightarrow f(x_n) < 0 \ \forall n \in \mathbb{N}$

$$\Rightarrow \underbrace{\lim_{n \rightarrow \infty} f(x_n)}_{\substack{\parallel \\ f(\hat{y})}} < 0. \quad \text{I.e. : } \boxed{f(\hat{y}) \leq 0} \quad \textcircled{*}_1$$

- Since $\hat{y} < b$, we can find a strictly decreasing sequence $(x'_n)_{n \in \mathbb{N}}$,

with $x'_n \xrightarrow[n \rightarrow \infty]{\hat{y}} \hat{y} \xrightarrow[f \text{ cont. at } \hat{y}]{\quad} f(x'_n) \xrightarrow[n \rightarrow \infty]{\quad} f(\hat{y})$.

$$\begin{aligned} \text{But } x'_n > \hat{y} \ \forall n \in \mathbb{N} &\Rightarrow f(x'_n) \geq 0 \ \forall n \in \mathbb{N} \\ \text{sup } A &\Rightarrow \underbrace{\lim_{n \rightarrow \infty} f(x'_n)}_{\substack{\parallel \\ f(\hat{y})}} \geq 0 \ \forall n \in \mathbb{N} \end{aligned}$$

I.e. : $\boxed{f(\hat{y}) \geq 0} \quad \textcircled{*}_2$

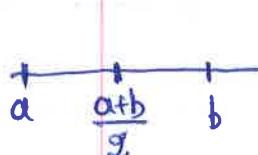
By $\textcircled{*}_1, \textcircled{*}_2$, we have: $f(\hat{y}) = 0$.

So, $\hat{y} \in (a, b)$ and $f(\hat{y}) = 0$. ■

(7)

2nd proof of Intermediate Value Theorem
(with nested intervals):

Split $[a, b]$ in two equal intervals, $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$.



- If $f\left(\frac{a+b}{2}\right) > 0$, then $f(a) < 0 < f\left(\frac{a+b}{2}\right)$.
- If $f\left(\frac{a+b}{2}\right) < 0$, then $f\left(\frac{a+b}{2}\right) < 0 < f(b)$.
- If $f\left(\frac{a+b}{2}\right) = 0$, then we have found $\exists z \in (a, b) : f(z) = 0$.

In each of the first two cases, f takes values with different signs at the endpoints of one of the intervals

$[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$. We name that interval $[a_1, b_1]$. Thus:

$$\textcircled{1} f(a_1) < 0 < f(b_1), \textcircled{2} b_1 - a_1 = \frac{b-a}{2}, \textcircled{3} [a_1, b_1] \subseteq [a, b].$$

Continuing this way, we either find $\exists z \in (a, b) : f(z) = 0$, or

nested closed intervals $[a, b] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots \supseteq [a_n, b_n] \supseteq \dots$

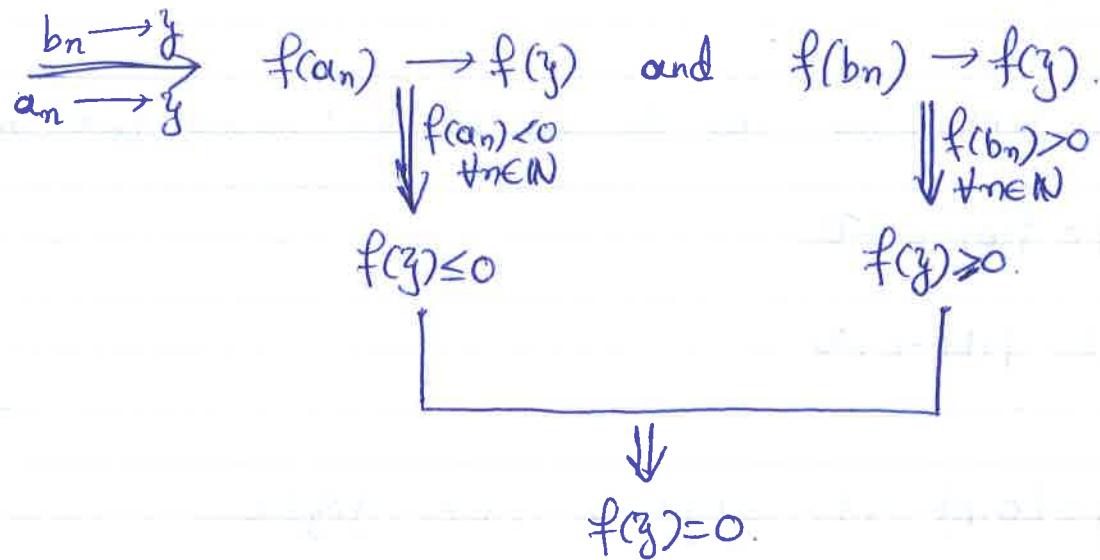
such that $f(a_n) < 0 < f(b_n) \quad \forall n \in \mathbb{N}$

$$\text{and } b_n - a_n = \frac{b-a}{2^n} \xrightarrow{n \rightarrow \infty} 0.$$

In this case, by the nested intervals theorem, we have:

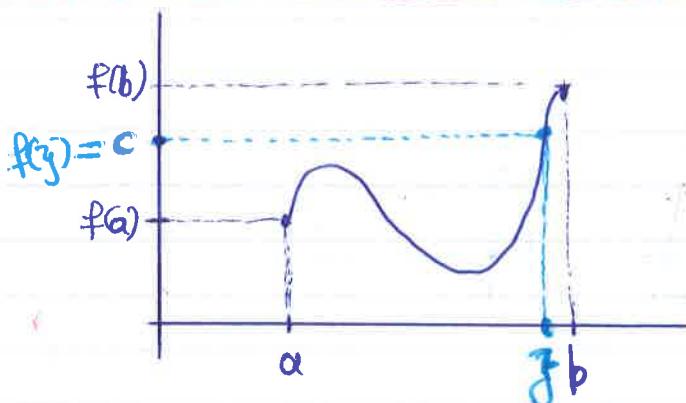
$$[a, b] \cap \left(\bigcap_{n=1}^{\infty} [a_n, b_n] \right) = \{z\}, \text{ where } z = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

In particular, $y \in [a, b] \Rightarrow f$ continuous at y



→ Other forms of the intermediate value theorem:

I Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous. Let $f(a) < c < f(b)$.
Then, there exists $y \in (a, b)$: $f(y) = c$.



(3)

Proof: Let $g(x) := f(x) - c$, $\forall x \in [a, b]$.

- g is continuous (as the sum of continuous functions)
 - $g(a) = f(a) - c < 0$.
 - $g(b) = f(b) - c > 0$
- } \Rightarrow

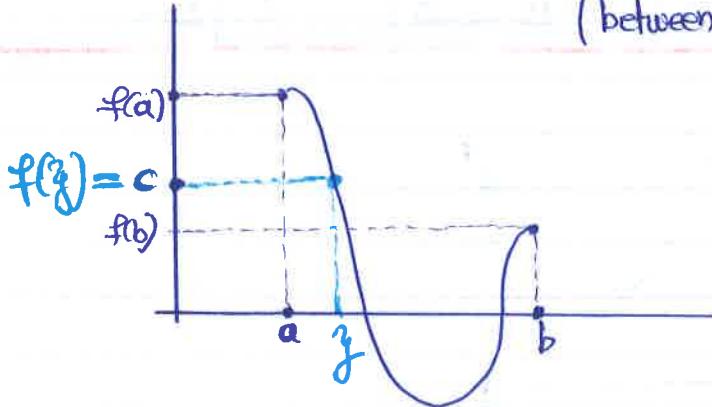
$\Rightarrow \exists y \in [a, b] \text{ s.t. } g(y) = 0$, i.e. $f(y) = c$.
 $\qquad\qquad\qquad \parallel$
 $\qquad\qquad\qquad f(y) - c$

(II)

Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous. Let $f(a) > c > f(b)$.

Then, there exists $y \in [a, b]$ s.t. $f(y) = c$.

(i.e.: f attains every value between two values!)



Proof: Let $f: [a, b] \rightarrow \mathbb{R}$

Let $g(x) := c - f(x)$, $\forall x \in [a, b]$.

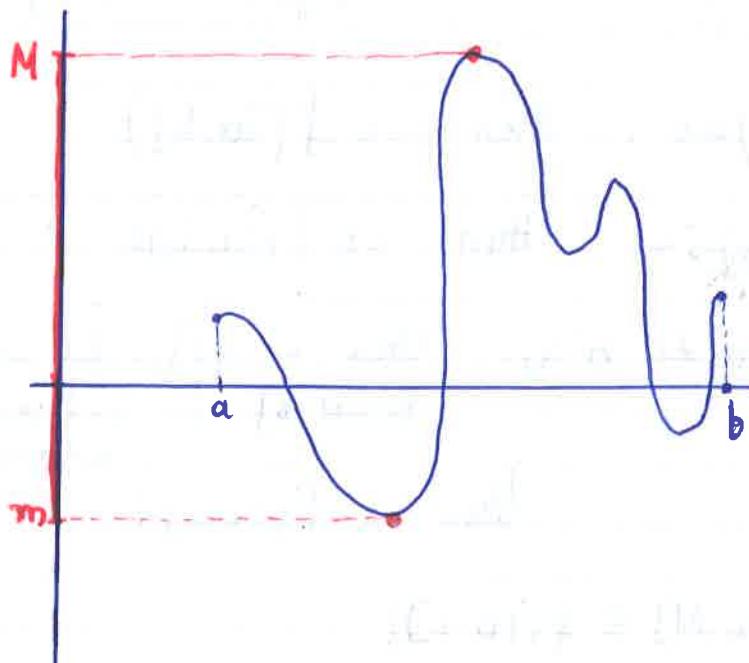
$\left. \begin{array}{l} \bullet g \text{ is continuous.} \\ \bullet g(a) = c - f(a) < 0. \\ \bullet g(b) = c - f(b) > 0 \end{array} \right\} \Rightarrow \exists z \in [a, b] \text{ s.t. } \underbrace{g(z)}_{c-f(z)} = 0, \\ \text{i.e. } f(z) = c. \blacksquare$

(III)

The continuous image of a closed interval is a closed interval. i.e.:

Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous. Then,

$$f([a, b]) = [m, M], \text{ for some } m \leq M \text{ in } \mathbb{R}.$$



Proof: Since $f: [a, b] \rightarrow \mathbb{R}$ continuous, we know that
 $\underbrace{[a, b]}_{\text{closed interval}}$

11

f attains a minimal and a maximal value in $[a,b]$.

I.e.: there exist $y_1, y_2 \in [a,b]$, s.t.

$$f(y_1) \leq f(x) \leq f(y_2), \quad \forall x \in [a,b].$$

In other words, for $m := f(y_1)$ and $M := f(y_2)$,

we have: $f([a,b]) \subseteq [m, M] \quad \textcircled{1}$

We will now show that: $[m, M] \subseteq f([a,b]) \quad \textcircled{2}$ WLOG, assume $y_1 \leq y_2$.

(Idea: f attains every value between m and M).

Indeed: let $c \in [m, M] \Leftrightarrow m \leq c \leq M$

$$\begin{matrix} \parallel \\ f(y_1) \end{matrix} \quad \begin{matrix} \parallel \\ f(y_2) \end{matrix}$$

- If $f(y_1) = c$, then $c \in f([a,b])$.
- If $f(y_2) = c$, then $c \in f([a,b])$.
- If $f(y_1) < c < f(y_2)$, then $c = f(\beta)$, for some $\beta \in [y_1, y_2]$ (as $f: [y_1, y_2] \rightarrow \mathbb{R}$ continuous), thus $c \in f([a,b])$.

Therefore, $[m, M] \subseteq f([a,b])$.

By $\textcircled{1}, \textcircled{2}$, $f([a,b]) = [m, M]$. ■

(12)

 In all these theorems, the assumption that f is continuous cannot be dropped. Similarly, the assumption that f is continuous on a closed interval cannot be dropped (Exercise!).

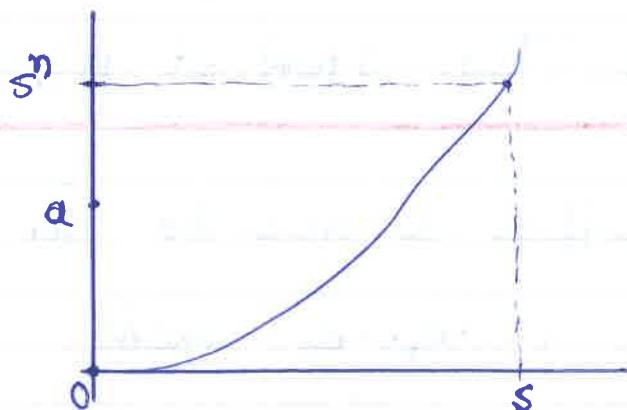
→ Applications of intermediate value theorem:

→ Each positive number has a unique n -th root. I.e.:

Let $a > 0$, $n \geq 2$. Then, there exists a unique $\beta > 0$,
s.t. $\beta^n = a$.

Proof: Let $f: [0, \infty) \rightarrow \mathbb{R}$, with $f(x) = x^n$.

We are asking for $\beta > 0$, s.t. $\beta^n = a$.



We will first find $s > 0$ s.t. $s^n > a$:

(13)

- If $a > 1$, then $a^n > a$ (so we pick $s=a$).
- If $a < 1$, then $1^n > a$ (so we pick $s=1$).
- If $a=1$, then $2^n > 1 = a$ (so we pick $s=2$).

(one can also say: $\lim_{x \rightarrow +\infty} x^n = +\infty$,
so $\exists s > 0$ s.t. $s^n > a$).

Now, $f(0) = 0 < a < s^n = f(s)$.

By the intermediate value theorem, there exists
 $y \in (0, s)$ s.t. $f(y) = a$, i.e. $y^n = a$.
(Uniqueness is easy).

(13)

- If $a > 1$, then $a^n > a$ (so we pick $s=a$).
- If $a < 1$, then $a^n > a$ (so we pick $s=1$).
- If $a=1$, then $2^n > 1 = a$ (so we pick $s=2$)
(one can also say: $\lim_{x \rightarrow +\infty} x^n = +\infty$,
so $\exists s > 0$ s.t. $s^n > a$).

Now, $f(0) = 0 < a < s^n = f(s)$.

By the intermediate value theorem, there exists $\xi \in (0, s)$ s.t. $f(\xi) = a$, i.e. $\xi^n = a$.
(Uniqueness is easy).

Lecture 20

①

→ Every polynomial of odd degree has a root in \mathbb{R}

I.e.: Let $n \in \mathbb{N}$ be odd, and p a polynomial of degree n . Then, $\exists \xi \in \mathbb{R}$ s.t. $p(\xi) = 0$.

Proof: It clearly suffices to show this for

$$p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0, \quad \forall x \in \mathbb{R}.$$

We have: $\lim_{x \rightarrow +\infty} (x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0) =$

(2)

suffices
to consider $x > 0$

$$= \lim_{x \rightarrow +\infty} \left[x^n \cdot \left(1 + \frac{a_{n-1}}{x} + \dots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right) \right] = +\infty$$

So, $\exists y > 0$ s.t. $p(y) > 0$.

Also, $\lim_{n \rightarrow +\infty} (x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0) =$

suffices to
consider $x < 0$

$$= \lim_{n \rightarrow +\infty} x^n \cdot \left(1 + \frac{a_{n-1}}{x} + \dots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right) = -\infty.$$

because n is odd!

So, $\exists y' < 0$ s.t. $p(y') < 0$.

Thus, we have found $y' < y$ in \mathbb{R} ,

with $p(y') < 0 < p(y)$.

Since $p: [y', y] \rightarrow \mathbb{R}$ is continuous,

by the intermediate value theorem there exists

$\exists z \in [y', y]$ s.t. $p(z) = 0$.

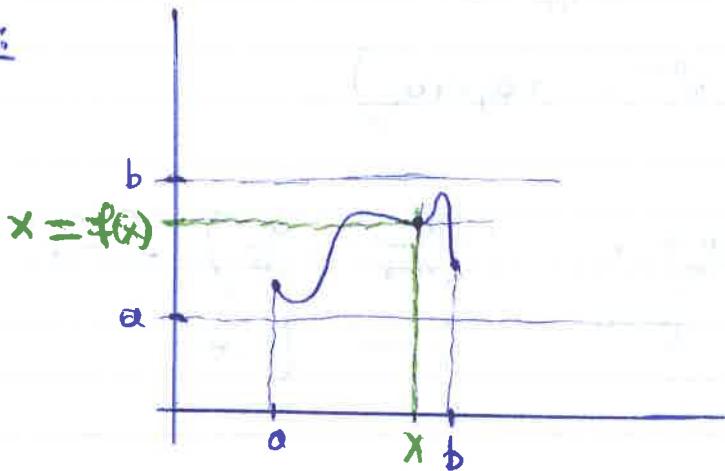
ex: $x^3 + 1$ has a real root, $x^2 + 1$ doesn't, $x^2 - 1$ does.

③

→ **fixed point theorem :** Let $f : [a, b] \rightarrow [a, b]$

be continuous. Then, there exists $x \in [a, b]$ s.t. $\underline{f(x)=x}$.

Proof:



Let $g : [a, b] \rightarrow [a, b]$, with $g(x) = f(x) - x$, $\forall x \in [a, b]$.

- g continuous (as f continuous).
- $g(a) = f(a) - a \geq 0$ (as $f([a, b]) \subseteq [a, b]$).
- $g(b) = f(b) - b \leq 0$ (as $f([a, b]) \subseteq [a, b]$).

If $g(a) = 0$, then $f(a) = a$.

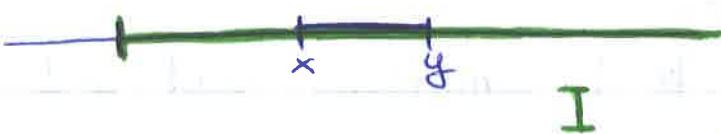
If $g(b) = 0$, then $f(b) = b$.

If $g(a) > 0 > g(b)$, then $\exists y \in (a, b)$ s.t. $g(y) = 0$,
i.e. s.t. $f(y) = y$.

(4)

→ Def: An $I \subseteq \mathbb{R}$ is called an interval if,

for all $x, y \in I$ with $x < y$, we have $[x, y] \subseteq I$.



It is easy to see that this definition is satisfied only for sets I that we have been calling intervals up to now:

$[a, b]$, $[a, b)$, $(a, b]$,
 (a, b) , $(a, +\infty)$, $[a, +\infty)$,
 $(-\infty, a)$, $(-\infty, a]$.

→ Continuous images of intervals are intervals: i.e.:

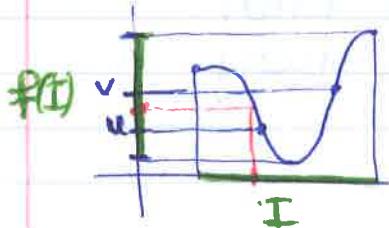
Let $f: I \rightarrow \mathbb{R}$ be continuous, where $I \subseteq \mathbb{R}$ is an interval. Then, $f(I)$ is an interval.

Proof: Let $u, v \in f(I)$, with $u < v$. We want to show:

$[u, v] \subseteq f(I)$. (Idea: f continuous, so it attains all values between u and v .)

Indeed, since $u, v \in f(I)$, we have that

$u = f(x)$ and $v = f(y)$, for some $x, y \in I$.



Let $z \in [u, v] \Rightarrow f(x) \leq z \leq f(y)$.

⑤

WLOG, assume $x \leq y$.
By the intermediate value theorem, there exists

$t \in [x, y] \stackrel{I}{\subbrace{}} \text{ s.t. } f(t) = z$. So, $z \in f(I)$

as I is an interval.

→ $(\text{Continuous on interval})$ and $(1-1)$ is monotone. i.e. :

Let $f: I \rightarrow \mathbb{R}$ be continuous and 1-1.
↓
an interval

Then, f is strictly monotone.

Proof: Let $u < v$ in I , fixed. Since f is 1-1,
we have $f(u) \neq f(v)$. So, either $f(u) < f(v)$,
or $f(u) > f(v)$.

Let us assume that $f(u) < f(v)$. We will
show that f is strictly increasing.

(and, similarly, if $f(u) > f(v)$
for the u, v we fixed,
then f would be strictly
decreasing).

(6)

Indeed: Let $x < y$ in I. We will show that $f(x) < f(y)$.

We will achieve this by showing that, if we put u, v, x, y in their correct order on the real line, then their images will be ordered in the same way. (and thus, since $x < y$, also $f(x) < f(y)$).

To prove this, we need an intermediate step:

If $a < b < c$ in I, then $f(a) < f(b) < f(c)$
or $f(a) > f(b) > f(c)$

Indeed:

Since f is 1-1, the values $f(a), f(b), f(c)$ are pairwise distinct.

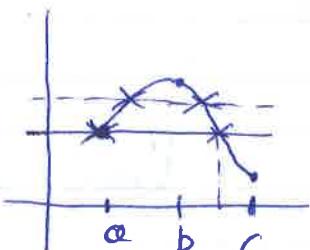
Now: either $f(a) < f(b)$ or $f(a) > f(b)$.

• Case 1: $f(a) < f(b)$:

- Suppose that $f(c) < f(a) < f(b)$

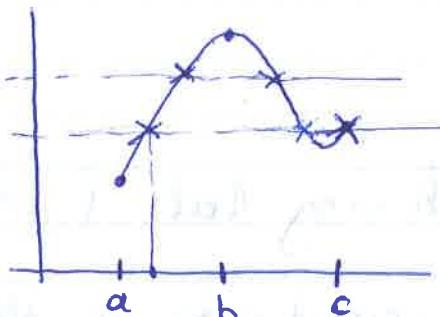
By the intermediate value theorem in $[b, c]$,

there exists $t \in [b, c]$ s.t. $f(t) = f(a) \Rightarrow f$ is not 1-1, contradiction.



7

- Suppose that $f(a) < f(c) < f(b)$.



By intermediate value

theorem on $[a, b]$,

$\exists t \in [a, b] \text{ s.t. } f(t) = f(c) \Rightarrow f \text{ not } 1-1$,
contradiction.

Thus: $f(a) < f(b) < f(c)$.

- Case 2: $f(a) > f(b)$:

Similarly, we see
that $f(a) > f(b) > f(c)$.

We thus deduce that:

If $t_1 < t_2 < t_3 < t_4$ in I, then either $f(t_1) < f(t_2) < f(t_3) < f(t_4)$
or $f(t_1) > f(t_2) > f(t_3) > f(t_4)$

(Indeed, if $f(t_1) < f(t_2)$, then $f(t_1) < f(t_2) < f(t_3)$

from the intermediate step, and, again by
this step, $f(t_2) < f(t_3) < f(t_4)$.

While, if $f(t_1) > f(t_2)$, then $f(t_1) > f(t_2) > f(t_3) > f(t_4)$.)

Now, we consider u, v, x, y ; we rename

the smallest of these a , the next in order
 b , the next c and the largest d ;
i.e., $a < b < c < d$.

We thus know that either

$$f(a) < f(b) < f(c) < f(d),$$

or $f(a) > f(b) > f(c) > f(d)$ \leadsto this is impossible,
as it would imply that
 $f(u) > f(v)$, a contradiction.

So, $f(a) < f(b) < f(c) < f(d)$.

In particular, since $x < y$, the above implies that
 $f(x) < f(y)$.

(8)

We thus know that either

$$f(a) < f(b) < f(c) < f(d),$$

or $f(a) > f(b) > f(c) > f(d)$ \leadsto this is impossible,
as it would imply that
 $f(u) > f(v)$, a contradiction.

$$\text{So, } f(a) < f(b) < f(c) < f(d).$$

In particular, since $x < y$, the above implies that
 $f(x) < f(y)$.

Lecture 21

(1)

Let $f: I \rightarrow \mathbb{R}$ be continuous and 1-1. Then:

↓
an interval

$f^{-1}: f(I) \rightarrow I$ is continuous, and has the same kind
of monotonicity as f . I.e. :

$$f \uparrow \Rightarrow f^{-1} \uparrow \quad \text{and} \quad f \downarrow \Rightarrow f^{-1} \downarrow$$

⚠ Remember : f continuous $\Rightarrow f(I)$ is an interval.
 f continuous + 1-1 $\Rightarrow f$ monotone
(strictly).

(2)

Proof:

- Suppose that f is strictly increasing.

Let $y_1 < y_2 \in f(I)$. We want: $f^{-1}(y_1) < f^{-1}(y_2)$.

Indeed, $\exists x_1, x_2 \in I$, s.t. $f(x_1) = y_1$ and $f(x_2) = y_2$.

$$f^{-1}(y_1) < f^{-1}(y_2)$$

If $x_1 > x_2$, then we would have $f(x_1) > f(x_2)$
(as $f \uparrow$), a contradiction.

If $x_1 = x_2$, then $f(x_1) \neq f(x_2)$ would

imply that f is not a function, a contradiction.

Thus, $x_1 < x_2$, i.e. $f(y_1) < f^{-1}(y_2)$.

Therefore, f^{-1} is strictly increasing.

(Similarly, we show that $f \downarrow \Rightarrow f^{-1} \uparrow$)

- f^{-1} is continuous:

Let $y_0 \in f(I)$. We will show that f^{-1}

is continuous at y_0 , in the case where

y_0 is not an endpoint of $f(I)$ (we work similarly if y_0 is an endpoint of $f(I)$)

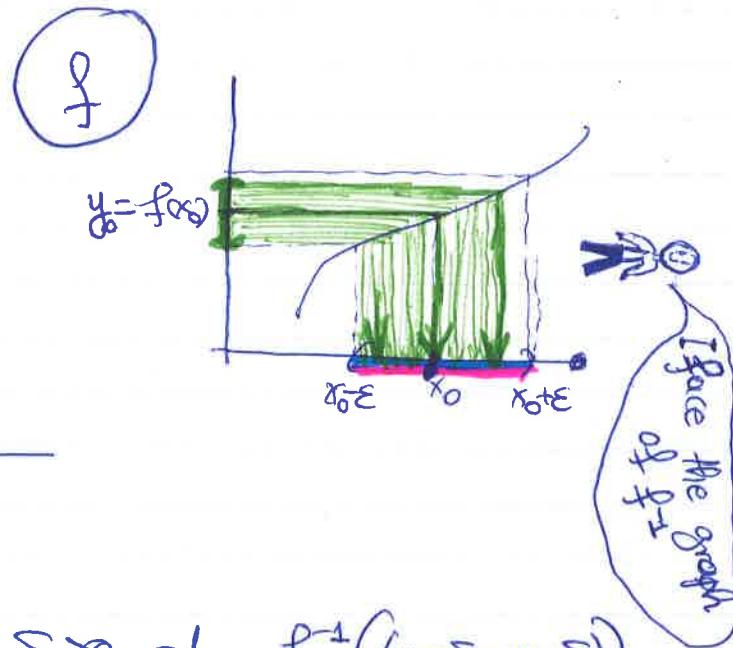
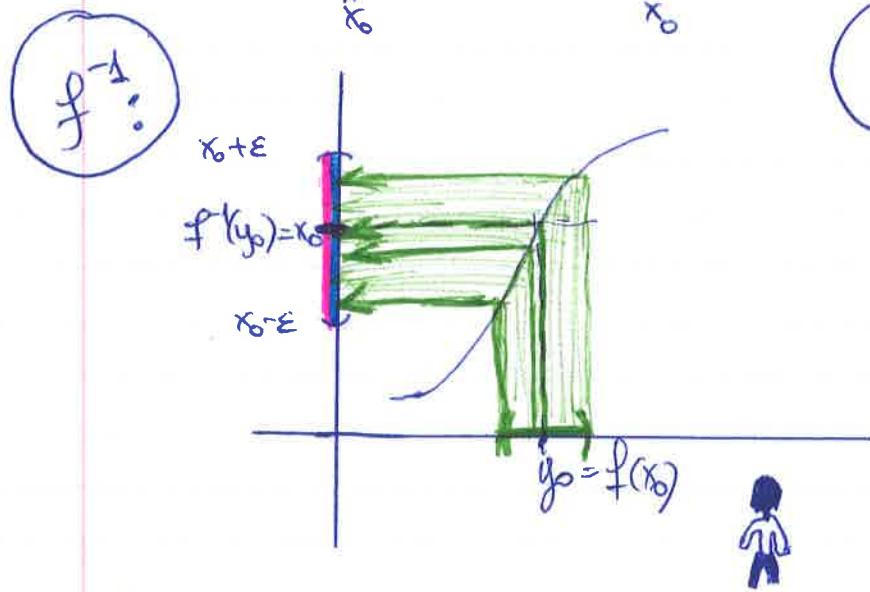
3

Let $f^{-1}(y_0) = x_0 (\in I)$.

Let $\varepsilon > 0$. We will show that the definition of continuity at y_0 is satisfied for f^{-1} for this $\varepsilon > 0$.

Note that we may assume that ε is small enough

for $(\underbrace{f^{-1}(y_0) - \varepsilon}_{x_0}, \underbrace{f^{-1}(y_0) + \varepsilon}_{x_0}) \subseteq I$



Goal: We are looking for $\delta > 0$ st. $f^{-1}((y_0 - \delta, y_0 + \delta)) \subseteq (x_0 - \varepsilon, x_0 + \varepsilon)$, i.e. st. $(y_0 - \delta, y_0 + \delta)$ is inverted inside $(x_0 - \varepsilon, x_0 + \varepsilon)$.

Idea: Since f is continuous and increasing, $f((x_0 - \varepsilon, x_0 + \varepsilon))$ is an interval containing y_0 . That interval is certainly inverted inside $(x_0 - \varepsilon, x_0 + \varepsilon)$!
(onto, actually).

(4)

We have: $x_0 - \varepsilon < x_0 < x_0 + \varepsilon$

$\overrightarrow{f \text{ is strictly increasing}}$

$$f(x_0 - \varepsilon) < f(x_0) < f(x_0 + \varepsilon).$$

So, for δ the minimal distance of $f(x_0 - \varepsilon)$ and $f(x_0 + \varepsilon)$ from $f(x_0)$

(i.e., for $\delta = \min \{f(x_0) - f(x_0 - \varepsilon), f(x_0 + \varepsilon) - f(x_0)\}$),

we have that,

$$\boxed{(*) \quad (f(x_0) - \delta, f(x_0) + \delta) \subseteq (f(x_0 - \varepsilon), f(x_0 + \varepsilon))}$$

(⚠ All that remains is to show that $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$ is the image of a neighbourhood of x_0)

Now: f is continuous, so it sends closed intervals to closed intervals. In particular,

$f([x_0 - \varepsilon, x_0 + \varepsilon])$ is a closed interval. So, since f is increasing, we have $f([x_0 - \varepsilon, x_0 + \varepsilon]) = [f(x_0 - \varepsilon), f(x_0 + \varepsilon)]$.

And therefore, since f is 1-1,

$$f((x_0 - \varepsilon, x_0 + \varepsilon)) = (f(x_0 - \varepsilon), f(x_0 + \varepsilon)).$$

By ⚡, $(y_0 - \delta, y_0 + \delta) \subseteq f((x_0 - \varepsilon, x_0 + \varepsilon))$,

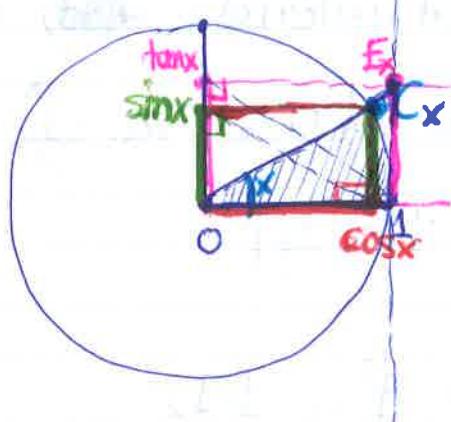
i.e., for this $\delta > 0$, $f^{-1}((y_0 - \delta, y_0 + \delta)) \subseteq (x_0 - \varepsilon, x_0 + \varepsilon)$. (5)

Since $\varepsilon > 0$ was arbitrary, f^{-1} is continuous at y_0 .

Since y_0 was an arbitrary element of the domain of f^{-1} , f^{-1} is continuous. ■

Trigonometric functions:

We define the trigonometric functions \sin , \cos , \tan as follows: Let $x \in \mathbb{R}$. We consider the



half-line with endpoint O and angle x with the positive x -axis. Let G_x be the point of intersection of this half-line with the unit circle centered at O . We define:

(b)

- $\cos x$ to be the projection of C_x on the x-axis.
- $\sin x$ to be the projection of C_x on the y-axis.

In other words, $\cos x, \sin x$ are the coordinates of C_x . I.e., $C_x = (\cos x, \sin x)$.

Since the length of OC_x is 1 (as C_x is on the unit circle), we have

$$\cos^2 x + \sin^2 x = 1, \forall x \in \mathbb{R}.$$

If the half-line OC_x (with endpoint O) intersects the vertical line $\{(x,y) : x=1\}$

(i.e., if $x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$),

we call the point of intersection E_x , and we define $\tan x$ to be the projection of E_x on the y-axis.

7

By similarity of triangles, we have

$$\frac{\tan x}{1} = \frac{\sin x}{\cos x}, \quad \text{i.e.}$$

$$\tan x = \frac{\sin x}{\cos x}$$

Let $E(x)$ be the area of the shaded cyclic sector (between 1 and C_x). The area of $E(x)$ is proportional to the angle x ,
 when $x \in [0, 2\pi]$. So:

$$\frac{x}{E(x)} = \frac{2\pi}{\text{area of unit disc}} = \frac{2\pi}{\pi \cdot 1^2} = 2$$

$$\rightarrow E(x) = \frac{x}{2},$$

$$\forall x \in [0, 2\pi].$$

(Notice that this proof works for $x \neq 0$; but, for $x=0$, $E(x)=0=\frac{x}{2}$ as well)

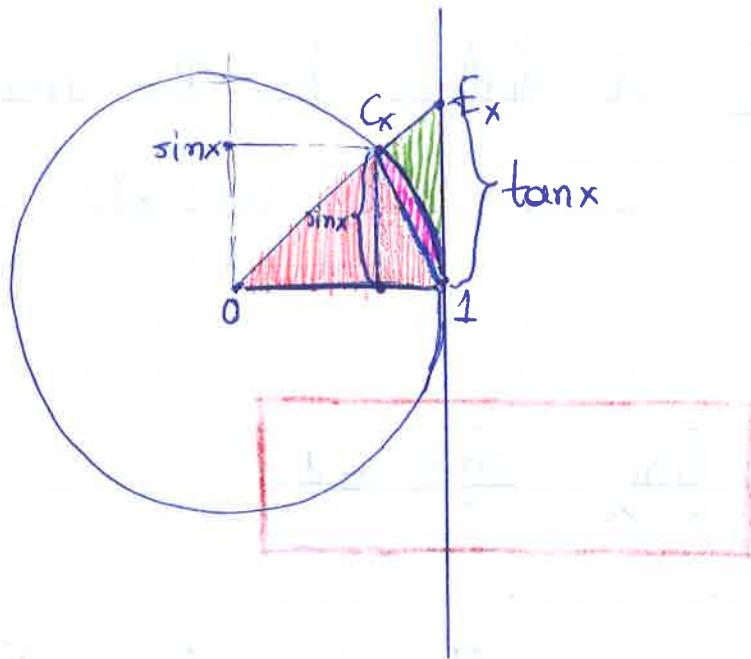
We use this fact to prove the following important inequality:

(8)

→ Prop: for $0 < x < \frac{\pi}{2}$, we have:

$$\sin x < x < \tan x.$$

Proof:



(area of triangle $C_x O 1$)

<

(area of cyclic sector $C_x O 1$)

<

(area of triangle $E_x O 1$), i.e.

$$\frac{1 \cdot \sin x}{2} < \underbrace{E(x)}_{\frac{x}{2}} < \frac{1 \cdot \tan x}{2}, \text{ i.e.}$$

$$\sin x < x < \tan x.$$

Obviously, this holds for the rest $x \in \mathbb{R}$ too; as, when $|x| \geq \frac{\pi}{2} \geq 1$, $|\sin x|$, which is ≤ 1 , is definitely smaller than $|x|$.

9

Corollary: If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then:

$$|\sin x| \leq |x|.$$

Proof: It follows by the previous inequality, using that $\sin(-x) = -\sin x$, $\forall x \in \mathbb{R}$. ■

Prop: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$

Proof: It suffices to show that

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1.$$

(Exercise).

for $\lim_{x \rightarrow 0^+} \frac{\sin x}{x}$: for $0 < x < \frac{\pi}{2}$, we have

$$\sin x < x < \tan x. \quad \text{And:}$$

- $\sin x < x \implies \frac{\sin x}{x} < 1.$

- $x < \tan x \implies x < \frac{\sin x}{\cos x} \implies \frac{\sin x}{x} > \cos x.$

(10)

So, $\forall x \in (0, \frac{\pi}{2})$:

$$\cos x < \frac{\sin x}{x} < 1$$

$\downarrow x \rightarrow 0^+$ $\downarrow x \rightarrow 0^+$

$$\cos 0 = 1$$

By the sandwich lemma, $\frac{\sin x}{x} \xrightarrow{x \rightarrow 0^+} 1$

Since $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$ $\forall x \neq 0$, we also have

$$\frac{\sin x}{x} \xrightarrow{x \rightarrow 0^-} 1. \quad \text{So, } \frac{\sin x}{x} \xrightarrow{x \rightarrow 0} 1.$$

There are two things we used that we haven't explained:

① The sandwich lemma holds for functions.

I.e.: If $g(x) \leq f(x) \leq h(x) \quad \forall x \neq x_0$,

$$\begin{matrix} \downarrow x \rightarrow x_0 \\ l \end{matrix} \quad \begin{matrix} \downarrow x \rightarrow x_0 \\ l \end{matrix}$$

then $f(x) \xrightarrow{x \rightarrow x_0} l$. This is left

as an exercise. (Do it using characterisation of limits via limits of sequences, + the sandwich lemma for sequences).

② $\cos x \xrightarrow[x \rightarrow 0]{} \cos 0 = 1$: This is because $\cos x$ is continuous at 0, and 0 is an accumulation point of the domain of \cos . We will explain this very soon.

In particular, to show that $\cos x$ is continuous, we first show that $\sin x$ is continuous (In fact, even better: $\sin x$ is Lipschitz continuous, which means that $|\sin x - \sin y| \leq M \cdot |x - y| \quad \forall x, y \in \mathbb{R}$, for some particular $M > 0$ (in this case, $M=1$ will work)). We will then have that $\cos x$ is continuous too, as $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, a composition of continuous functions.

So mainly the "hard" part is to show that $\sin x$ is continuous. To that end, we will need two trigonometric identities. (the last 2 in the list that follows).

→ **Trigonometric identities:** If $a, b \in \mathbb{R}$, we have:

- (1) $\cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b$
- (2) $\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$
- (3) $\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b$
- (4) $\sin(a-b) = \sin a \cdot \cos b - \cos a \cdot \sin b$

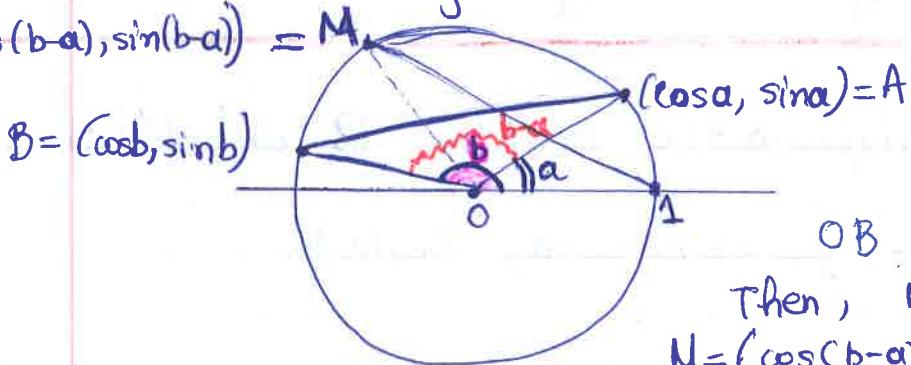
Proof: Notice how (2), (3) and (4) just follow from (1):

(1) → (2) as $\cos(a+b) = \cos(a - (-b))$, so we can apply (1) for the real numbers $a, -b$.

(1) → (3), as $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ $\forall x \in \mathbb{R}$, so
again, this follows from (1) $\sin(a+b) = \cos\left(\frac{\pi}{2} - a - b\right)$

And then (3) → (4), as $\sin(a-b) = \sin(a + (-b))$.

Thus, it only suffices to show (1):



Rotate the triangle BOA around O, so that OA becomes O1, OB becomes OM, for some M. Then, $\hat{NO1} = \hat{BO1} = b - a$, so $M = (\cos(b-a), \sin(b-a))$.

Moreover, (length of line segment AB)

=

(length of line segment M1),

$$\text{i.e. } \left| (\cos b - \cos a, \sin b - \sin a) \right|^2 = \left| (\cos(b-a) - 1, \sin(b-a)) \right|^2,$$

$$\text{i.e. } (\cos b - \cos a)^2 + (\sin b - \sin a)^2 = (\cos(b-a) - 1)^2 + (\sin(b-a))^2$$

$$\Leftrightarrow \underbrace{\cos^2 b + \cos^2 a - 2 \cos b \cos a}_{= \cos^2(b-a)} + \underbrace{\sin^2 b + \sin^2 a - 2 \sin b \sin a}_{= \sin^2(b-a)} =$$

$$= \underbrace{\cos^2(b-a) + 1 - 2 \cos(b-a)}_{\cos^2(b-a)} + \underbrace{\sin^2(b-a)}_{\sin^2(b-a)}$$

$$\Leftrightarrow \cancel{2} - 2 \cos a \cos b - 2 \sin a \sin b = \cancel{2} - 2 \cos(b-a) \Leftrightarrow$$

$$\Leftrightarrow \cos(b-a) = \cos a \cos b - \sin a \sin b.$$



Corollary: It holds that

$$\sin x - \sin y = 2 \cdot \sin \frac{x-y}{2} \cos \frac{x+y}{2}, \quad \forall x, y \in \mathbb{R}.$$

Proof: By the trigonometric identities (3) and (4) above,

* $\sin(a+b) - \sin(a-b) = 2 \sin b \cos a, \quad \forall a, b \in \mathbb{R}.$

Let $x, y \in \mathbb{R}$. Set $a = \frac{x+y}{2}$, $b = \frac{x-y}{2}$ above; we get

$a+b=x$ and $a-b=y$, thus

$$\textcircled{*} \Rightarrow \sin x - \sin y = 2 \sin \frac{x-y}{2} \cos \frac{x-y}{2}.$$

→ Prop: $\sin : \mathbb{R} \rightarrow \mathbb{R}$ is continuous (Lipschitz continuous, in fact).

Proof: Let $x, y \in \mathbb{R}$.

Then, by the Corollary above,

$$|\sin x - \sin y| = 2 \cdot \left| \sin \frac{x-y}{2} \right| \cdot \left| \cos \frac{x-y}{2} \right| \leq$$

$$\underbrace{\left| \sin \frac{x-y}{2} \right|}_{\leq \left| \frac{x-y}{2} \right|} \cdot \underbrace{\left| \cos \frac{x-y}{2} \right|}_{\leq 1}$$

$\leq 2 \cdot \frac{|x-y|}{2} = |x-y|$. Thus, \sin is Lipschitz continuous, and therefore continuous (you have proved that Lipschitz continuous \Rightarrow continuous in Weekly Assignment 6).

→ Corollary: $\cos : \mathbb{R} \rightarrow \mathbb{R}$ is continuous

Proof: $\cos x = \sin \left(\frac{\pi}{2} - x \right) \forall x \in \mathbb{R}$. Thus, \cos is

continuous, as the composition of continuous functions.

→ Prop: $\cos: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous.

Proof: As for \sin , but using trigonometric identities (1) and (2). Exercise.